Sharp bounds on the local lemma

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In this report, we review the Lovász local lemma in the symmetric case and give sharp estimates for the required condition by following Shearer (1985). In Section 1, we give a proof of the (symmetric) Lovász local lemma, and see that the famous condition '$ep(d+1) \le 1$' can be weakened to '$epd \le 1$' when the dependency graph is undirected. In Section 2, we see that the bound we prove in Section 1 is actually optimal, by constructing a counterexample for the range of parameter violating the bound.

Although this report mostly followed the techniques given in the paper Shearer (1985), we have rearranged and reformulated the original argument to make it easier to understand and made several remarks explaining the intuition behind the proofs.

1 Sharper local lemma for undirected graphs

In combinatorics, probabilistic methods are used for proving the existence of certain objects. An abstract way of proving the existence of an object satisfying the property P is to simply consider a randomized object X and prove $\mathbb{P}(X \text{ satisfies } P) > 0$. When the property P can be broken down into simpler properties $P_1, \ldots, P_n$ so that $P_1 \wedge \cdots \wedge P_n$ implies P, then we can use a union bound $\mathbb{P}(X \text{ satisfies } P) \ge \mathbb{P}(X \text{ satisfies } P_1 \wedge \cdots \wedge P_n) \ge 1 - \sum_{i=1}^n \mathbb{P}(X \text{ violates } P_i)$, which shows the desired existence when $\sum_{i=1}^n \mathbb{P}(X \text{ violates } P_i) < 1$. However, in many situations, we can obtain a cleverer bound than this by using additional information of X and P_i's.

In the following, we denote the event that X satisfies P_i by A_i and assume $\mathbb{P}(A_i) > 0$. Our objective is to establish a general scheme for evaluating $\mathbb{P}\left(\bigcap_{i=1}^n A_i\right)$. For example, if these events $A_1, \ldots, A_n$ are independent, we simply have $\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n \mathbb{P}(A_i) > 0$. The Lovász local lemma (Erdős & Lovász 1973, Spencer 1977) can treat a more general form of independence.

Let $[n] := \{1, \ldots, n\}$ for a positive integer n. A directed graph G with vertices $[n]$ is called *dependency digraph* for $(A_i)_{i=1}^n$ if the event A_i is independent from the family $(A_j)_{j \ne i, i \not\to j}$ for each $i = 1, \ldots, n$. We discuss the 'symmetric' version of the local lemma in this report:

Theorem 1 (Local lemma)**.** *Let G be a dependency digraph for the events $A_1, \ldots, A_n$ whose out-degrees are all at most d. If, for some $p \in [0,1)$, $\mathbb{P}(A_i^c) \le p$ holds for each $i = 1, \ldots, n$ and $ep(d+1) \le 1$ holds, then we have $\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) > 0$.*

Proof. We first prove that for each $i \in [n]$ and $J \subset [n] \setminus \{i\}$ we have $\mathbb{P}\left(A_i^c \mid \bigcap_{j \in J} A_j\right) \le 1/(d+1)$ inductively on $|J|$. Note that we regard $\bigcap_{j \in \emptyset} A_j = \Omega$, where Ω is the whole sample space. So this holds true if $|J| = 0$ as $\mathbb{P}(A_i^c) \le p \le 1/(e(d+1))$. Now assume the inequality holds for all cases with smaller cardinality of J. Let $J_1 = \{j \in J \mid i \to j\}$ and $J_2 = J \setminus J_1$. Then, we have

$$\mathbb{P}\left(A_i^c \mid \cap_{j \in J} A_j\right) = \mathbb{P}\left(A_i^c \middle| \bigcap_{j \in J_1} A_j \cap \bigcap_{k \in J_2} A_k\right) = \frac{\mathbb{P}\left(A_i^c \cap \bigcap_{j \in J_1} A_j \mid \bigcap_{k \in J_2} A_k\right)}{\mathbb{P}\left(\bigcap_{j \in J_1} A_j \mid \bigcap_{k \in J_2} A_k\right)}$$

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by the Bayes rule. The numerator can be bounded by the independence as

$$\mathbb{P}\left(A_i^c \cap \bigcap_{j\in J_1} A_j \,\middle|\, \bigcap_{k\in J_2} A_k\right) \le \mathbb{P}\left(A_i^c \,\middle|\, \bigcap_{k\in J_2} A_k\right) = \mathbb{P}\left(A_i^c\right) \le p \le \frac{1}{e(d+1)}.$$

Let $J_1 = \{j_1, \dots, j_m\}$, where $m = |J_1| \le d$. Then, by writing $J_1(\ell) = \{j_q \mid q < \ell\}$ for $\ell = 1, \dots, m$, the denominator can be evaluated as

$$\mathbb{P}\left(\bigcap_{j\in J_1} A_j \mid \bigcap_{k\in J_2} A_k\right) = \prod_{\ell=1}^{m} \mathbb{P}\left(A_{j_\ell} \,\middle|\, \bigcap_{j\in J_1(\ell)\cup J_2} A_j\right) \ge \left(1 - \frac{1}{d+1}\right)^m \ge \left(1 + \frac{1}{d}\right)^{-d} \ge \frac{1}{e},$$

where we have used the induction hypothesis (if J_1 is empty the denominator is simply 1). Therefore, we have proven $\mathbb{P}\left(A_i^c \mid \bigcap_{j\in J} A_j\right) \le 1/(d+1)$ for $i \notin J$. Then, we can prove the desired fact as

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}\left(A_i \,\middle|\, \bigcap_{j\in[i-1]} A_j\right) \ge \left(1 - \frac{1}{d+1}\right)^n > 0,$$

where $[0] = \emptyset$. □

Although this is the standard form of the local lemma widely used in the literature, we can actually loosen the condition $ep(d+1) \le 1$ to $epd \le 1$ *when G is undirected*, i.e., $i \to j \iff j \to i$ (Shearer 1985). We call such a graph just a *dependency graph* in this note. Indeed, the usual construction of the dependency digraph is done by finding independent random variables $Y_1, \dots, Y_N$ so that each A_i is $\sigma(Y_k, k \in K_i)$-measurable for some $K_i \subset [N]$, and make $i \to j$ if and only if $K_i \cap K_j = \emptyset$. As this construction gives an undirected graph (in the sense $i \to j$ if and only if $j \to i$), the condition $epd \le 1$ is sufficient in practice.

Before going into details of the stronger bound, let us decipher the proof given above. Indeed, it suffices to prove $\mathbb{P}\left(A_i \mid \bigcap_{j\in[i-1]} A_j\right) > 0$, or more strictly $\mathbb{P}\left(A_i \mid \bigcap_{j\in J} A_j\right) \ge \lambda > 0$ uniformly. What we have done then is the following evaluation (with the notation in the proof):

$$\mathbb{P}\left(A_i \,\middle|\, \bigcap_{j\in J} A_j\right) = 1 - \mathbb{P}\left(A_i^c \,\middle|\, \bigcap_{j\in J} A_j\right) \ge 1 - \frac{\mathbb{P}\left(A_i^c\right)}{\prod_{\ell=1}^{m} \mathbb{P}\left(A_{j_\ell} \mid \bigcap_{j\in J_1(\ell)\cup J_2} A_j\right)}. \qquad (*)$$

Thus, if λ satisfies $\lambda = 1 - p/\lambda^d$, the induction works. If we set $f(\lambda) = \lambda^d - \lambda^{d+1}$, then f takes its maximum at $\lambda = d/(d+1)$ with $f(\lambda) = d^d/(d+1)^{d+1}$, so it gives the above proof. As is already mentioned, in the case of undirected dependency graph, we can refine the proof to get the following bound.

Theorem 2 (Local lemma for undirected graphs; Shearer 1985)**.** *Let G be a dependency graph for the events $A_1, \dots, A_n$ whose degrees are all at most d. If, for some $p \in [0, 1)$, $\mathbb{P}\left(A_i^c\right) \le p$ holds for each $i = 1, \dots, n$ and $epd \le 1$ holds, then we have $\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) > 0$.*

Remark 3. As we shall see in the following, $p \le (d-1)^{d-1}/d^d$ ($> 1/(ed)$) is sufficient for $d \ge 2$ and $p < 1/2$ for $d = 1$ (equality sensitive). The latter is trivial as the events have independent decomposition into pairs or singletons.

Proof of Theorem 2. It suffices to prove $\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) > 0$ when $d \ge 2$ and $\mathbb{P}\left(A_i^c\right) \le p := (d-1)^{d-1}/d^d$ holds uniformly for all i. Let $\lambda = (d-1)/d$, which satisfies $\lambda = 1 - p/\lambda^{d-1}$. We first prove the following claim:

> **Claim.** Let $i \in [n]$ and $J \subset [n] \setminus \{i\}$. If the number of vertices in J connected to i is at most $d-1$, then we have $\mathbb{P}\left(A_i \mid \cap_{j\in J} A_j\right) > \lambda$.

Let us prove this claim by induction on $|J|$. If $|J| = 0$ it is true as $\mathbb{P}(A_i) \geq 1 - p \geq 1 - p/\lambda^{d-1} = \lambda$. For the general case, let $J_1 = \{j_1, \dots, j_m\} \subset J$ (where $m \leq d-1$) be the vertices connected to i, and $J_2 = J \setminus J_1$. Let also $J_1(\ell) = \{j_q \mid q < \ell\}$ for $\ell = 1, \dots m$. By using the inequality $(*)$ Note that j_ℓ *is connected to at most* $d-1$ *points in* $J_1(\ell) \cup J_2$ (as it is already connected to i), so $\mathbb{P}\left(A_{j_\ell} \mid \bigcap_{j \in J_1(\ell) \cup J_2} A_j\right) > \lambda$ for each ℓ by induction hypothesis. Therefore, we obtain, from the inequality $(*)$,

$$\mathbb{P}\left(A_i \,\middle|\, \bigcap_{j\in J} A_j\right) \geq 1 - \frac{\mathbb{P}(A_i^c)}{\prod_{\ell=1}^m \mathbb{P}\left(A_{j_\ell} \mid \bigcap_{j\in J_1(\ell)\cup J_2} A_j\right)} > 1 - \frac{p}{\lambda^m} \geq 1 - \frac{p}{\lambda^{d-1}} = \lambda.$$

So the induction works.

Consider the pair $i \in [n]$ and $J \subset [n] \setminus \{i\}$ such that the degree of i is d in $J \cup \{i\}$. In this case, by using the same argument (except $m = d$) as above, we can prove $\mathbb{P}\left(A_i \mid \bigcap_{j\in J} A_j\right) > 1 - p/\lambda^d$. Indeed, $1 - p/\lambda^d = 1 - 1/\lambda + (1 - p/\lambda^{d-1})/\lambda = 2 - 1/\lambda \geq 0$ as we have $\lambda \leq 1/2$ for $d \geq 2$. In particular, we have proven $\mathbb{P}\left(A_i \mid \bigcap_{j\in A_j}\right) > 0$ for all the cases. Therefore, we have $\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n \mathbb{P}\left(A_i \mid \bigcap_{j\in[i-1]} A_j\right) > 0$. □

2 Optimality

More surprising result of Shearer (1985) is it has completely determined the upper bound of such p. For $d \geq 2$, define $p_0(d)$ be the supremum of p satisfying the following condition:

Condition A. *For an arbitrary* n *and events* $A_1, \dots, A_n$ *with a dependency graph* G *whose degrees are all at most* d, $\mathbb{P}(A_i^c) \leq p$ *for all* $i = 1, \dots, n$ *implies* $\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) > 0$.

As we shall see, it actually coincides with the bound in Remark 3, i.e., $p_0(d) = (d-1)^{d-1}/d^d$. Therefore, the *optimal* bound for the local lemma in the undirected case has been specified.

Let us discuss the construction of a counterexample when $(d-1)^{d-1}/d^d < p \leq 1$. For a given graph $G = (V, E)$ with degrees at most d, let 2^V be the set of all subsets of V, and $I(G) \subset 2^V$ be the set of all independent sets of G, i.e., $S \subset I(G)$ if and only if S does not contain vertices i, j who are adjacent in G. Consider the following function $Q_G : 2^V \to \mathbb{R}$:

- $Q_G(S) = 0$ if $S \notin I(G)$.
- $\sum_{T\supset S} Q_G(T) = \sum_{T\supset S,\, T\in I(G)} Q_G(T) = p^{|S|}$ if $S \in I(G)$.

This Q_G is indeed well-defined. The value $Q_G(S)$ is equal to $p^{|S|}$ for a maximal independent set S, and we can inductively determine $Q_G(S) = p^{|S|} - \sum_{T\supsetneq S} Q(T)$, and it does not cause any conflict as $I(G)$ has a poset structure. By using the inclusion-exclusion principle, we have another expression

$$Q_G(S) = \sum_{T\supset S,\, T\in I(G)} (-1)^{|T|-|S|} p^{|T|}.$$

Remark 4. Let us explain some intuition behind this. If $Q_G(S)$ is nonnegative for all $S \subset 2^V$, then Q_G is actually a probability measure as $\sum_S Q_G(S) = \sum_{S\supset\emptyset} Q_G(S) = 1$. Then we can consider a family of events $(A_v)_{v\in V}$ with

$$\mathbb{P}\left(\bigcap_{v\in S} A_v^c \cap \bigcap_{w\in V\setminus S} A_w\right) = Q_G(S)$$

for each $S \subset V$. Then, note that for each $S \in I(G)$ we have $\mathbb{P}\left(\bigcap_{v\in S} A_v^c\right) = \sum_{T\subset S} Q_G(S) = p^{|S|}$. G is then a dependency graph for these events. Indeed, for $v \in V$ and $S \in I(G)$ not adjacent to v, we have, since $S \cup \{v\} \in I(G)$,

$$\mathbb{P}\left(A_v^c \cap \bigcap_{w\in S} A_w^c\right) = p^{1+|S|} = \mathbb{P}\left(A_v^c\right)\mathbb{P}\left(\bigcap_{w\in S} A_w^c\right).$$

The same conclusion holds even when $S \notin I(G)$ as long as S is not adjacent to v (in that case both sides becomes zero), so by an application of π-λ theorem, A_v is independent from $\sigma(A_w, (v,w) \notin E)$. Therefore, the function Q_G is a way to construct a family of events satisfying (1) the probability of each event is $1-p$, (2) G is a dependency graph for the events, provided Q_G is nonnegative everywhere.

We can consider the function Q also for subgraphs of G. For a subset of vertices $W \subset V$, let $G(W) = (W, E|_{V\times V})$ be the induced subgraph of G and we just denote by $Q_{G(W)}$ the function defined for this graph (for simplicity, $Q_{G(\emptyset)}$ is a constant function with $Q_{G(\emptyset)}(\emptyset) = 1$). Then, we can prove the following lemma[1].

Lemma 5. *If $Q_G(\emptyset) < 0$ holds, there exists a subset $W \subset V$ such that*

$$Q_{G(W)}(\emptyset) < 0, \qquad Q_{G(W)}(S) \geq 0 \quad \textit{for all} \quad S \in 2^W \setminus \{\emptyset\}.$$

Proof. Since $Q_G(\emptyset) < 0$, we can find a minimal $W \subset V$ with $Q_{G(W)}(\emptyset) < 0$, i.e., $Q_{G(W)}(\emptyset) < 0$ and $Q_{G(U)}(\emptyset) \geq 0$ for all $U \subsetneq W$. We shall prove that this W actually satisfies the desired property.

Assume $Q_{G(W)}(S) < 0$ for some nonempty $S \subset W$. Let $U \subset W \setminus S$ be the vertices in $W \setminus S$ that are not adjacent to S. Then, we have

$$\begin{aligned} Q_{G(W)}(S) &= \sum_{T\supset S,\, T\in I(G(W))} (-1)^{|T|-|S|} p^{|T|} \\ &= p^{|S|} \sum_{U\in I(G(U))} (-1)^{|U|} p^{|U|} = p^{|S|} Q_{G(U)}(\emptyset). \end{aligned}$$

This implies $Q_{G(U)}(\emptyset) < 0$ and contradicts the minimality of W. So $Q_{G(W)}(S) \geq 0$ holds for all $\emptyset \subsetneq S \subset W$. ☐

If we are given such a subset W, $Q_{G(W)}$ *almost* gives a probability measure over 2^W. Let $R_{G(W)}(S) = \sum_{S\subset T\subset W} Q_{G(W)}(T)$. Note that from the very first definition of Q_G, we have

$$R_{G(W)}(\emptyset) = \sum_{T\subset W} Q_{G(W)}(T) = 1, \qquad R_{G(W)}(\{v\}) = \sum_{\{v\}\subset T\subset W} Q_{G(W)}(T) = p$$

for each $v \in W$. From the same argument in Remark 4, for $v \in W$ and $S \subset W \setminus \{v\}$ not adjacent to v, we also have

$$R_{G(W)}(S \cup \{v\}) = pR_{G(W)}(S). \tag{†}$$

We shall obtain a probability measure over 2^W by modifying $Q_{G(W)}$ slightly. Find $\delta \in (0,1)$ such that $\delta Q_{G(W)}(\emptyset) + (1-\delta)(1-p)^{|W|} = 0$ (or $\delta = (1-p)^{|W|}/((1-p)^{|W|} - Q_{G(W)}(\emptyset))$ explicitly). Then, consider the function $\tilde{Q} : 2^W \to \mathbb{R}$ given by

$$\tilde{Q}(S) = \delta Q_{G(W)}(S) + (1-\delta)p^{|S|}(1-p)^{|W|-|S|}, \quad S \subset W.$$

[1] In the original paper (Shearer 1985), the author only assumes $Q_G(S) < 0$ for some $S \subset V$ to deduce the same conslusion, but we could not prove this so give a surrogate statement here.

This is a 'mixture' of the signed measure $Q_{G(W)}$ and the product of Bernoulli measures. From the definition of δ, and the fact $\sum_{S\subset W} p^{|S|}(1-p)^{|W|-|S|} = (p+(1-p))^{|W|} = 1$, this is actually a probability measure over 2^W. So, we can consider a family of events $(A_v)_{v\in W}$ with

$$\mathbb{P}\left(\bigcap_{v\in S} A_v^c \bigcap_{w\in W\setminus S} A_w\right) = \tilde{Q}(S), \quad S\subset W.$$

Here, for each $S\subset W$,from the definition of $\tilde{Q}$, we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{v\in S} A_v^c\right) &= \sum_{S\subset T\subset W} \tilde{Q}(T) = \delta R_{G(W)}(S) + (1-\delta)\sum_{S\subset T\subset W} p^{|T|}(1-p)^{|W|-|T|} \\ &= \delta R_{G(W)}(S) + (1-\delta)p^{|S|}.\end{aligned}$$

Therefore, by using (†), for $v\in W$ and $S\subset W\setminus\{v\}$ not adjacent to v, we have

$$\mathbb{P}\left(A_v^c \cap \bigcap_{w\in S} A_w^c\right) = p(\delta R_{G(W)}(S) + (1-\delta)p^{|S|}) = \mathbb{P}\left(A_v^c\right)\mathbb{P}\left(\bigcap_{w\in S} A_w^c\right).$$

Again, by the use of π-λ theorem, we see that $G(W)$ is actually a dependency graph for the events $(A_v)_{v\in W}$. Now, notice that by the definition of δ we have $\mathbb{P}\left(\bigcap_{v\in W} A_v\right) = \tilde{Q}(\emptyset) = 0$. As $G(W)$ still is a graph of degree at most d, this p violates Condition A. By combining with Lemma 5, we have proven the following statement.

Lemma 6. *Let G be a graph with degrees at most d. For an arbitrary $p\in(0,1]$, if $Q_G(\emptyset) = \sum_{S\in I(G)}(-p)^{|S|} < 0$ holds, then p violates Condition A.*

The final step for showing $p_0(d) = (d-1)^{d-1}/d^d$ is to construct a graph satisfying the assumption of Lemma 6 for each $p > (d-1)^{d-1}/d^d$. For a given $d\geq 2$, define a sequence of rooted trees $(T_m)_{m=0}^\infty$ as follows:

- T_0 is of a single vertex, which is also a root.
- For $m\geq 0$, construct T_{m+1} by connecting a new root x with the roots of $d-1$ copies of T_m.

If $d=3$, this is just the construction of a complete binary tree of depth m. It is clear that every vertex in T_m is of degree at most d.

Let us consider the independent sets in T_{m+2}. Let T_{m+2} consist of the root x and $T_{m+1}^{(1)},\ldots,T_{m+1}^{(d-1)}$ (copies of T_{m+1}) with roots $x^{(1)},\ldots,x^{(d-1)}$. Then, an independent set $S\in I(T_{m+2})$ is either given by $\bigcup_{i=1}^{d-1} S^{(i)}$ with $S^{(i)}\in I(T_{m+1}^{(i)})$, or $\{x\}\cup\bigcup_{i=1}^{d-1}\tilde{S}^{(i)}$ with $\tilde{S}^{(i)}\in I(T_{m+1}^{(i)}\setminus\{x^{(i)}\})$, where $T_{m+1}^{(i)}\setminus\{x^{(i)}\}$ means the induced subgraph of $T_{m+1}^{(i)}$ without $x^{(i)}$ and is of $d-1$ disjoint copies of T_m. Therefore, if we simply denote $a_m = Q_{T_m}(\emptyset)$, we have

$$\begin{aligned}a_{m+2} &= \sum_{S\in I(T_{m+2})}(-p)^{|S|} = \left(\sum_{S\in I(T_{m+1})}(-p)^{|S|}\right)^{d-1} - p\left(\sum_{S\in I(T_m)}(-p)^{|S|}\right)^{(d-1)^2} \\ &= a_{m+1}^{d-1} - p a_m^{(d-1)^2}\end{aligned}$$

for $m\geq 0$. Also, we can see $a_0 = 1-p$ and $a_1 = a_0^{d-1} - p = (1-p)^{d-1} - p$. We want to know for which values of p the sequence $(a_m)_{m=0}^\infty$ becomes negative at some m. If $a_m > 0$ and $a_{m+1} = 0$ then $a_{m+2} < 0$, so it suffices to consider the condition of a_m being positive for all m.

We follow the technique of Shearer (1985) hereafter. Let $b_m = a_{m+1}/a_m^{d-1}$ for $m\geq 0$. Then, $b_{m+1} = a_{m+2}/a_{m+1}^{d-1} = 1 - p/b_m^{d-1}$. So, for $0<p<1$, the positivity of $(a_m)_{m=0}^\infty$ is equivalent to the positivity of the sequence given by $b_{-1} = 1$ and $b_m = 1 - p/b_{m-1}^{d-1}$ for $m \geq 0$. If $b_m > 0$ for all m, $b_{m+1} - b_m = p(b_{m-1}b_m)^{1-d}(b_m^{d-1} - b_{m-1}^{d-1})$, so from $b_{-1} > b_0$ inductively $b_0 > b_1 > b_2 > \cdots > 0$ holds. Thus, there exists a limit $\lambda = \lim_{m\to\infty} b_m$. Note that $0 < \lambda < 1$, as $b_{m+1} = 1 - p/b_m^{d-1}$ becomes negative if b_m is sufficiently small. Therefore, the limit satisfies $\lambda = 1 - p/\lambda^{d-1}$.

In other words, for a value of p such that there is no $0 < \lambda < 1$ satisfying $\lambda = 1 - p/\lambda^{d-1}$, or equivalently $p = \lambda^{d-1} - \lambda^d$, a_m becomes negative and consequently p violates Condition A. Similarly from what we have seen briefly before Theorem 2, the maximum value of $f(\lambda) = \lambda^{d-1} - \lambda^d$ for $\lambda \in (0, 1)$ is $(d-1)^{d-1}/d^d$. Therefore, we have the following conclusion:

Theorem 7. *For each $d \geq 2$, $p \in (0, 1)$ satisfies Condition A if and only if $p \leq (d-1)^{d-1}/d^d$.*

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