# Lie groups and Lie algebras

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In this report, we provide a brief introduction to the Lie theory, especially Lie's third theorem, which asserts that every finite-dimensional Lie algebra can be represented as a Lie algebra of a Lie group. Although we have mainly followed the flow of arguments in Kobayashi & Oshima (2005), as the authors introduce Lie groups as those locally isomophic to linear Lie groups, we have largely modified details referencing Carter et al. (1995) and Duistermaat & Kolk (2000). Note that, though we only consider the case of real Lie groups and Lie algebras for simplicity, there are parallel results for the complex case.

In Section 1, we describe the overview of the theory, and in what logic Lie's third theorem is established. In Section 2, we give the definition of a Lie group and its Lie algebra. We prove that Lie algebras captures the local structure of Lie groups (Theorem 1) in Section 3. Finally, in the case of the general linear group $GL(n, \mathbb{R})$ we show that any Lie subalgebra is a Lie algebra of some Lie subgroup of $GL(n, \mathbb{R})$, which is the special case of Theorem 4.

To deepen my understanding, I have tried to make this report close to self-contained and give alternative proofs in my own way if possible. Because of this approach, however, the arguments have become rather long, and I could not include some important aspects such as representation theoretic viewpoints.

## 1 Introduction

In this section, we explain the big picture of the Lie theory informally. In this theory, there are two kinds of objects: Lie groups and Lie algebras. To provide an overview of the theory, we sometimes use a word without definition in this section. The detailed explanation is given in the following sections.

A *Lie group* is a smooth manifold equipped with a group structure whose operations are smooth. such examples have been used widely. the set of nonzero real numbers $\mathbb{R}^\times$ is a commutative Lie group with its multiplication; the space of all the $n \times n$ invertible matrices $GL(n, \mathbb{R})$ is an example of non-commutative Lie group with matrix multiplication; the orthogonal group $O(n)$ of $A \in GL(n, \mathbb{R})$ such that $A^\top A = I_n$ is a compact Lie group. For each Lie group $G$, we can define a so-called *Lie algebra* of $G$, which we shall denote by $\mathrm{Lie}(G)$. Usually $\mathrm{Lie}(G)$ is defined by adding a bracket operation $[\cdot, \cdot]$ on the vector space $T_1G$, which is the tangent space of $G$ at the unit element 1. The Lie algebra of a Lie group $G$ indeed determines the local structure of $G$ in the following sense:

**Theorem 1.** *Lie groups $G$ and $G'$ are locally isomorphic if and only if the corresponding Lie algebras $\mathrm{Lie}(G)$ and $\mathrm{Lie}(G')$ are isomorphic.*

This assertion is significant in that the geometric study of the local structure of a Lie group can be transferred into the algebraic study of its Lie algebra.

Apart from the concept of Lie algebras of Lie groups, we also have abstract Lie algebras, the definition of which has no relation to Lie groups: a (real) Lie algebra $\mathfrak{g}$ is a real vector space equipped

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with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called *Lie bracket* that satisfies

$$[x,x] = 0, \qquad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

for any $x, y, z \in \mathfrak{g}$. If we have a real associative algebra $A$, i.e., $A$ is a real vector space equipped with a ring structure that is compatible to the scalar multiplication, we can confirm that the commutator $[x,y] := xy - yx$ becomes a Lie bracket on $A$ by a simple comuputation. Hence, $\mathbb{R}$, $M(n,\mathbb{R})$ (the space of all $n \times n$ real matrices) and $\mathfrak{o}(n)$ (the space of all skew-symmetric $n \times n$ real matrices) are Lie algebras with the commutator. These abstract Lie algebras are indeed isomorphic to the Lie algebras of the Lie groups $\mathbb{R}^\times$, $GL(n,\mathbb{R})$ and $O(n)$, respectively.

Remarkably, Lie's third theorem assures that in general there is always such a Lie group for each finite-dimensional Lie algebra:

**Theorem 2** (Lie's third theorem)**.** *Any finite-dimensional Lie algebra is isomorphic to a Lie algebra of a Lie group.*

The proof of this theorem is difficult, but can be done by exploiting the following purely algebraic result, which we do not prove in this note.

**Theorem 3** (Ado's theorem)**.** *Any finite-dimensional Lie algebra is isomorphic to a Lie subalgebra of $M(n,\mathbb{R})$ for some positive integer $n$.*

In this note, our main objective is to derive Lie's third theorem from Ado's theorem. To do so, we prove the following theorem:

**Theorem 4** (analytic subgroup)**.** *Let $G$ be a Lie group. For any Lie subalgebra $\mathfrak{h}$ of $\mathrm{Lie}(G)$, there is a unique connected Lie subgroup $H \subset G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$.*

Indeed, once we establish Theorem 4, and the fact the Lie algebra of $GL(n,\mathbb{R})$ is $M(n,\mathbb{R})$, it follows that any Lie subalgebra of $M(n,\mathbb{R})$ is a Lie algebra of some Lie subgroup of $GL(n,\mathbb{R})$. Then Lie's third theorem immediately follows from Ado's theorem. It is also notable that any Lie group is locally isomorphic to a Lie subgroup of $GL(n,\mathbb{R})$, which follows from Theorem 1 and Theorem 4 applied to Ado's theorem.

# 2 Lie algebra of a Lie group

We start with basic definitions and properties of Lie groups.

**Definition 5.** A Lie group is a smooth manifold $G$ equipped with a group structure, such that the mappings

$$G \times G \to G; (x,y) \mapsto xy, \qquad G \to G; x \mapsto x^{-1}$$

are smooth.

Before proceeding to the definition of $\mathrm{Lie}(G)$, let us define tangent spaces and vector fields of a smooth manifold without using the local coordinates. For a smooth manifold $M$, let $C^\infty(M)$ be the set of all smooth maps $M \to \mathbb{R}$.

**Definition 6.** Let $M$ be a smooth manifold. The tangent space of $M$ at $p \in M$ is defined as

$$T_pM := \left\{ v : C^\infty(M) \to \mathbb{R} \;\middle|\; \begin{array}{c} v \text{ is linear,} \\ v(fg) = f(p)v(g) + g(p)v(f) \end{array} \right\}.$$

Also, a linear map $X : C^\infty(M) \to C^\infty(M); f \mapsto X(f)$ is called a vector field on $M$ if $X(\cdot)(p) \in T_pM$ for each $p \in M$.

By using a local coordinate $x = (x_1, \dots, x_n)$ around $p$, we can prove that each $v \in T_pM$ has a form $v = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ for some $a_i \in \mathbb{R}$. We can also prove that each vector field $X$ locally has a form

$$X = \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i},$$

where each $b_i$ is smooth. Let us denote by $\mathfrak{X}(M)$ the set of all the vector fields on $M$. Then, for $X, Y \in \mathfrak{X}(M)$, as a map $C^\infty(M) \to C^\infty(M)$, define $[X, Y] := f \mapsto X[Y[f]] - Y[X[f]]$. Then, for $f, g \in C^\infty(M)$, we have

$$X(Y(fg)) = X(fY(g) + gY(f)) = X(f)Y(g) + X(g)Y(f) + fX(Y(g)) + gX(Y(f)),$$

and so

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f) \qquad \therefore\ [X, Y] \in \mathfrak{X}(M).$$

This implies that $\mathfrak{X}(M)$ is a Lie algebra in the abstract sense.

**Definition 7.** For a Lie group $G$, define

$$\mathrm{Lie}(G) := \{X \in \mathfrak{X}(G) \mid \pi_g X = X \pi_g,\ \forall g \in G\},$$

where $\pi_x : C^\infty(G) \to C^\infty(G)$ for $x \in G$ is defined by $\pi_x(f)(y) = f(x^{-1}y)$.

It is clear that $\mathrm{Lie}(G)$ is closed under the Lie bracket as $\pi_g XY = X\pi_g Y = XY\pi_g$ holds for $X, Y \in \mathrm{Lie}(G)$. We next prove that $\mathrm{Lie}(G)$ is regarded as $T_1(G)$.

**Proposition 8.** *For any Lie group $G$, $\mathrm{Lie}(G)$ and $T_1(G)$ are isomorphic as a real vector space.*

*Proof.* Let $X \in \mathrm{Lie}(G)$. Then, by the definition of vector fields, $X_1 := X(\cdot)(1) \in T_1(G)$. Then, from the commutativity $\pi_x X = X\pi_x$, we have, for any $f \in C^\infty(G)$ and $x \in G$,

$$(Xf)(x) = (\pi_{x^{-1}} Xf)(1) = (X\pi_{x^{-1}} f)(1) = X_1(\pi_{x^{-1}} f),$$

so $X$ is determined by $X_1$ and the map $\iota : \mathrm{Lie}(G) \to T_1(G); X \mapsto X_1$ is injective. It is clear that $\iota$ is a linear map, so it suffices to prove its surjectivity. Let $v \in T_1(G)$ and define $X^v$ as

$$X^v(f)(x) := v(\pi_{x^{-1}} f), \qquad f \in C^\infty(G),\ x \in G.$$

The linearity of $X^v$ is clear as $v$ is a linear map. The smoothness of $v(\pi_{x^{-1}} f)$ with respect to $x$ follows from the fact that $v$ is indeed a differential operator at 1. We also have

$$\begin{aligned} X^v(fg)(x) &= v(\pi_{x^{-1}} fg) = v((\pi_{x^{-1}} f)(\pi_{x^{-1}} g)) \\ &= (\pi_{x^{-1}} f)(1) v(\pi_{x^{-1}} g) + (\pi_{x^{-1}} g)(1) v(\pi_{x^{-1}} f) \\ &= f(x) X^v(g)(x) + g(x) X^v(f)(x), \end{aligned}$$

so $X^v$ is indeed a vector field. Since $\iota(X^v) = v$ holds, $\iota$ is surjective and threfore a vector space isomorphism. □

From this proposition, we see that $\mathrm{Lie}(G)$ is a finite-dimensional vector space, whose dimension is the same as the manifold dimension of $G$.

We next construct the exponential map $\exp : \mathrm{Lie}(G) \to G$. For an $X \in \mathrm{Lie}(G)$, consider the expression

$$X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$$

using the local coordinate $x = (x_1, \dots, x_n)$ around $1 \in G$. Then, consider a local ODE for $c : \mathbb{R} \to G$ given by

$$\frac{\mathrm{d}c_i(t)}{\mathrm{d}t} = a_i(c(t)), \tag{1}$$

where $c(t) = (c_1(t), \ldots, c_n(t))$ in the local coordinate. Then, there exists a unique solution $c(t)$ such that $c(0) = 1$ in some interval $(-\varepsilon, \varepsilon)$. Note that the ODE is equivalent to $c(t)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) = (Xf)(c(t)) \tag{2}$$

for all $f \in C^\infty(G)$ by chain rule. If $|s|$ is sufficiently small, from $\pi_{x^{-1}} X = X \pi_{x^{-1}}$ with $x = c(s)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(c(s)c(t)) = (Xf)(c(s)c(t)).$$

As we also have $\frac{\mathrm{d}}{\mathrm{d}t} f(c(s+t)) = (Xf)(c(s+t))$ with the same initial value $c(s)$, from the uniqueness of ODE's solution, we have $c(s+t) = c(s)c(t)$, i.e., $c$ is locally a homomorphism. For $t$ outside the interval $(-\varepsilon, \varepsilon)$, we can define $c(t) = c(t/n)^n$ for a sufficiently large $n$. This is well-defined as $c$ is a homomorphism. For this $c$ defined all over $\mathbb{R}$, the ODE (2) is globally satisfied for any $f \in C^\infty(G)$. We in particular denote $c(1)$ by $e^X$ and call it the exponential map. It is clearly well-defined: $\exp tX = c(t)$ holds from the uniqueness of ODE's solution. We may also write $e^X$ as $\exp X$.

Note that for each $f \in C^\infty(G)$, $x \in G$ and $Y \in \mathrm{Lie}(G)$, we have, from the repeated use of (2),

$$(Y^k f)(xe^{tY}) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(xe^{tY}), \qquad k = 0, 1, \ldots, \ t \in \mathbb{R}.$$

Therefore, generally for $Y_1, \ldots, Y_m \in \mathrm{Lie}(G)$, we have

$$\begin{aligned}
(Y_1^{k_1} \cdots Y_m^{k_m} f)(x) &= \frac{\partial^{k_1}}{\partial t_1^{k_1}} (Y_2^{k_2} \cdots Y_m^{k_m} f)(xe^{t_1 Y_1}) \Big|_{t_1=0} \\
&= \frac{\partial^{k_1}}{\partial t_1^{k_1}} \frac{\partial^{k_2}}{\partial t_2^{k_2}} (Y_3^{k_3} \cdots Y_m^{k_m} f)(xe^{t_1 Y_1} e^{t_2 Y_2}) \Big|_{t_2=t_1=0} \\
&= \cdots = \frac{\partial^{k_1}}{\partial t_1^{k_1}} \cdots \cdots \frac{\partial^{k_m}}{\partial t_m^{k_m}} f(xe^{t_1 Y_1} \cdots e^{t_m Y_m}) \Big|_{t_m=\cdots=t_1=0}.
\end{aligned}$$

This gives the Taylor expansion of the function $(t_1, \ldots, t_m) \mapsto f(xe^{t_1 Y_1} \cdots e^{t_m Y_m})$:

$$f(xe^{t_1 Y_1} \cdots e^{t_m Y_m}) \sim \sum_{k_1, \ldots, k_m \geq 0} \frac{t_1^{k_1}}{k_1!} \cdots \frac{t_m^{k_m}}{k_m!} (Y_1^{k_1} \cdots Y_m^{k_m} f)(x). \tag{3}$$

The map exp has the following property:

**Proposition 9.** *For any Lie group $G$, the exponential map* $\exp : \mathrm{Lie}(G) \to G$ *is a smooth map.*

*Proof.* Take a basis $\{X_1, \ldots, X_n\}$ of $\mathrm{Lie}(G)$. For an $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$, define $Y_s := s_1 X_1 + \cdots + s_n X_n$. Take a local coordinate $x = (x_1, \ldots, x_n)$ around $1 \in G$. As each $X_i$ is a differential operator, it has an expression

$$X_i = a_{ij}(x) \frac{\partial}{\partial x_j}, \qquad j = 1, \ldots, n,$$

where each $a_{ij}$ is smooth. Then, $\exp(tYs)$ locally satisfies the ODE (from (1))

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tY_s)_j = \sum_{i=1}^n s_i a_{ij}(\exp(tY_s)).$$

As an ODE is smooth with respect to the change of parameters, the mapping $(t, s) \mapsto \exp(tY_s)$ is smooth in a neighborhood of 0. By fixing a sufficiently small $t$, we have $s \mapsto Y_s$ is smooth in a neighborhood of 0. Denote the image of this neighborhood by $U$.

Then, as $U$ is a neighborhood of $0 \in \mathrm{Lie}(G)$, for any $X \in \mathrm{Lie}(G)$, there exists a positive integer $n$ such that $U$ is also a neighborhood of $n^{-1}X$. Hence, we have that

$$Y \mapsto e^Y = \underbrace{e^{n^{-1}Y} \cdots e^{n^{-1}Y}}_{n\text{times}}$$

is smooth on a neighborhood of $X$. This means that exp is smooth everywhere. □

In particular, as the derivative exp at 0 is identity, we have that exp is a diffeomorphism between some neighborhood $U$ of $0 \in \mathrm{Lie}(G)$ and $V$ of $1 \in G$. Let us denote the smooth inverse map of exp by $\log : V \to U$.

The following assertion assures that we can make $G$ *analytic* (Duistermaat & Kolk 2000). We do not prove this theorem here. We assume some metric on $\mathrm{Lie}(G)$ induced by its Euclidean-space structure.

**Theorem 10.** *Let $G$ be a Lie group. The map $(X, Y) \mapsto \log(e^X e^Y)$ is analytic on $U_0 \times U_0$ for some neighborhood $U_0$ of $\mathrm{Lie}(G)$. Moreover, we can choose $U_0$ so that for $V_0 := \exp(U_0)$, the $\kappa^x : xV_0 \to U_0$ given by*

$$\kappa^x(y) = \log(x^{-1}y)$$

*forms an analytic atlas of $G$.*

More formally, the transition map $\kappa^x \circ (\kappa^y)^{-1}$ is analytic on $\kappa^y(xV_0 \cap yV_0)$ for each $x, y \in G$, and the operations of $G$ are analytic in terms of this atlas. Remark that the composition of analytic maps is also analytic, and the inverse function theorem holds with analyticity (Krantz & Parks 2002).

# 3 Correspondence up to local isomorphism

Hereafter, we assume all Lie groups are equipped with analytic structure given by their exponential maps. The following proposition is essential in proving Theorem 1.

**Proposition 11.** *Let $G$ be a Lie group. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathrm{Lie}(G)$. Then, for $x \in G$, the mapping $\Psi_x : \mathrm{Lie}(G) \to G$ given by*

$$\Phi_x(t_1X_1 + \cdots + t_nX_n) = xe^{t_1X_1} \cdots e^{t_nX_n}, \qquad (t_1, \ldots, t_n) \in \mathbb{R}^n$$

*attains an analytic diffeomorphism between some neighborhoods of $0 \in \mathrm{Lie}(G)$ and $x \in G$.*

*Proof.* From the definition of $\kappa^x$, we have

$$(\kappa^x \circ \Phi_x)(t_1X_1 + \cdots + t_nX_n) = \log(e^{t_1X_1} \cdots e^{t_nX_n}).$$

If we denote the mapping $(X, Y) \mapsto \log(e^X e^Y)$ by $\mu$, we have that the $n$-fold composition

$$\mu^n := \underbrace{\mu(\cdot, \mu(\cdot, \ldots \mu(}_{n \text{ times}}\cdot, \cdot) \ldots))$$

is analytic on $\tilde{U}^n$ for a sufficiently small neighborhood $\tilde{U}$ of $0 \in \mathrm{Lie}(G)$. As $\kappa^x \circ \Psi_x = \mu^n$ holds, we have, from analyticity,

$$\log(e^{t_1X_1} \cdots e^{t_nX_n}) = t_1X_1 + \cdots + t_nX_n + \mathrm{O}\big(t_1^2 + \cdots + t_n^2\big)\,, \tag{4}$$

where we equip $\mathrm{Lie}(G)$ with a norm of $\|t_1X_1 + \cdots + t_nX_n\|^2 = t_1^2 + \cdots + t_n^2$. Hence, by the inverse function theorem, this $\kappa^x \circ \Phi_x$ is locally an analytic diffeomorphism. As $(\kappa^x)^{-1}$ is by definition an analytic diffeomorphism, $\Phi_x$ also admits the property. □

We next prove an algebraic useful result.

**Proposition 12.** *Let $G$ be a Lie algebra, and let $\{X_1, \ldots, X_n\}$ be the basis of* $\mathrm{Lie}(G)$. *Consider, for each positive integer $k$,*

$$\mathfrak{D}_k(G) := \mathrm{span}\{X_1^{i_1} \cdots X_n^{i_n} \mid i_1 + \cdots + i_n \leq k\},$$

*as a linear subspace of the space of all left-invariant differential operators. Then, for any $Y_1, \ldots, Y_k \in$* $\mathrm{Lie}(G)$, *$Y_1 \cdots Y_k \in \mathfrak{D}_k(G)$ holds.*

*Proof.* We prove both statements by induction on $k$. The case $k = 1$ is obvious. Then, from the induction hypothesis and the linearity, it suffices to prove

$$X_1^{i_1} \cdots X_n^{i_n} X_\ell \in \mathfrak{D}_k(G)$$

for each $i_1 + \cdots + i_n = k - 1$ and $\ell \in \{1, \ldots, n\}$. As we have

$$X_j^{i_j} X_\ell = X_\ell X_j^{i_j} + \sum_{p=0}^{i_j - 1} X_j^p [X_j, X_\ell] X_j^{i_j - p - 1},$$

for each $j$, we obtain

$$\begin{aligned} X_1^{i_1} \cdots X_n^{i_n} X_\ell &= X_1^{i_1} \cdots X_{\ell-1}^{i_{\ell-1}} X_\ell^{i_\ell + 1} X_{\ell+1}^{i_{\ell+1}} \cdots X_n^{i_n} \\ &\quad + \sum_{q=\ell+1}^{n} \sum_{p=0}^{i_j - 1} X_1^{i_1} \cdots X_{j-1}^{i_{j-1}} X_j^p [X_j, X_\ell] X_j^{i_j - p - 1} X_{j+1}^{i_{j+1}} \cdots X_n^{i_n}, \end{aligned}$$

and the conclusion holds as each summand has only $k-1$ terms (recall $[X_j, X_\ell]$ is a linear combination of $\{X_1, \ldots, X_n\}$). ☐

Now recall Theorem 1:

> *Lie groups $G$ and $G'$ are locally isomorphic if and only if the corresponding Lie algebras* $\mathrm{Lie}(G)$ *and* $\mathrm{Lie}(G')$ *are isomorphic.*

We say $G$ and $G'$ are locally isomorphic if for some neighborhood $U$ of $1 \in G$ there is a homeomorphism $\varphi : U \to \varphi(U)$ such that $\varphi(U)$ is a neighborhood of $1 \in G'$ and

$$xy \in U \Leftrightarrow \varphi(x)\varphi(y) \in \varphi(U), \qquad xy \in U \Rightarrow \varphi(x)\varphi(y) = \varphi(xy).$$

Also, by stating $\mathrm{Lie}(G)$ and $\mathrm{Lie}(G')$ are isomorphic, we mean there is a vector space isomorphism $T$ preserving the Lie bracket: $T([X, Y]) = [T(X), T(Y)]$ for $X, Y \in \mathrm{Lie}(G)$.

*Proof of Theorem 1.* (*if* part) First we only consider $G$. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathrm{Lie}(G)$. As $\Phi_1$ defined in Proposition 11 is analytic and $G$ itself has an analytic structure,

$$F : (s, t) = ((s_1, \ldots, s_n), (t_1, \ldots, t_n)) \mapsto e^{s_1 X_1} \cdots e^{s_n X_n} e^{t_1 X_1} \cdots e^{t_n X_n}$$

is an analytic map. For sufficiently small $s$ and $t$, again from Proposition 11, there exists an analytic function $u = (u_1, \ldots, u_n)$ of $(s, t)$ such that $F(s, t) = e^{u_1 X_1} \cdots e^{u_n X_n}$.

For any analytic function $f$ on a (sufficiently small) neighborhood of $1 \in G$, from (3), we have

$$\begin{aligned} f(F(s,t)) &= \sum_{i_1, \ldots, i_n, j_1, \ldots, j_n \geq 0} \frac{s_1^{i_1} \cdots s_n^{i_n} t_1^{j_1} \cdots t_n^{j_n}}{i_1! \cdots i_n! j_1! \cdots j_n!} (X_1^{i_1} \cdots X_n^{i_n} X_1^{j_1} \cdots X_n^{j_n} f)(1) \\ &= \sum_{k_1 \ldots k_n \geq 0} \frac{u_1^{k_1} \cdots u_n^{k_n}}{k_1! \cdots k_n!} (X_1^{k_1} \cdots X_n^{k_n} f)(1). \end{aligned}$$

From Proposition 12, there are constants such that

$$X_1^{i_1}\cdots X_n^{i_n}X_1^{j_1}\cdots X_n^{j_n} = \sum_{|k|\le|i|+|j|} a_{i,j,k}X_1^{k_1}\cdots X_n^{k_n},$$

where $|\alpha| := \alpha_1+\cdots+\alpha_n$ for $\alpha\in\mathbb{Z}_{\ge 0}^n$. Note that thi $a_{i,j,k}$ can be determined by only the coefficients of $[X_e,X_f]$ in terms of the basis $\{X_g\}$ (indeed the proof of Proposition 12 gives an algorithm).

Then, for the locally analytic function $f : e^{u_1X_1}\cdots e^{u_nX_n}\mapsto u_\ell$, we have

$$t_\ell = \sum_{i,j} C_{i,j,\delta^\ell}\frac{s^{i_1}\ldots s^{i_n}t^{j_1}\cdots t^{j_n}}{i_1!\cdots i_n!j_1!\cdots j_n!}, \tag{5}$$

where $(\delta^\ell)_i = 1_{\{i=\ell\}}$.

As the Lie algebras are isomorphic, exactly the same coefficients $C_{i,j,k}$ appear in the $G'$ counterpart. Therefore, the group multiplication operations (determined by $\Phi_1$ and $F$) of $G$ and $G'$ are locally homeomorphic.

(*only if* part) Let $\varphi$ be the local hommeomorphism between $G$ and $G'$. For $X\in\mathrm{Lie}(G)$ and sufficiently small $|s|,|t|$, we have $\varphi(e^{(s+t)X}) = \varphi(e^{sX}e^{tX}) = \varphi(e^{sX})\varphi(e^{tX})$. From the continuity, there must exist a unique $X'\in\mathrm{Lie}(G')$ such that $\varphi(e^{tX}) = e^{tX'}$ (in a neighborhood, and indeed globally). As we can also consider $\varphi^{-1}$, this correspondence between $X$ and $X'$ is bijective. We denote it by $\iota:\mathrm{Lie}(G)\to\mathrm{Lie}(G')$. We shall prove that $\iota$ is indeed a isomorphism.

From (4), for $X,Y\in\mathrm{Lie}(G)$ and sufficiently small $|t|$, we have

$$m\log\left(\exp\left(\frac{t}{m}X\right)\exp\left(\frac{t}{m}Y\right)\right) = t(X+Y)+\mathrm{O}\left(\frac{t^2}{m}\right),$$

so by continuity $\lim_{m\to\infty}(e^{\frac{t}{m}X}e^{\frac{t}{m}Y})^m = e^{t(X+Y)}$. As we have $\varphi(e^{\frac{t}{m}X}e^{\frac{t}{m}Y}) = e^{\frac{t}{m}\iota(X)}e^{\frac{t}{m}\iota(Y)}$, we obtain $\iota(X+Y) = \iota(X)+\iota(Y)$. Together with the sclar multiplication, $\iota$ is linear.

It only remains to prove that $\iota([X,Y]) = [\iota(X),\iota(Y)]$. We prove this by using (3). By letting $x=1$, $m=4$, $t_i=t$, $(Y_1,Y_2,Y_3,Y_4) = (X,Y,-X,-Y)$, we can compute that

$$f(e^{tX}e^{tY}e^{-tX}e^{-tY}) = f(1)+t^2([X,Y]f)(1)+\mathrm{O}(t^3).$$

Then, as $[X,Y]\in\mathrm{Lie}(G)$, from the very definition of exp in (2), we have, by letting $f=\log$ locally,

$$([X,Y]\log)(1) = [X,Y]\log\exp(t[X,Y])|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\log\exp(t[X,Y])\bigg|_{t=0} = [X,Y]. \tag{6}$$

Hence we see that $\log(e^{tX}e^{tY}e^{-tX}e^{-tY}) = t^2[X,Y]+\mathrm{O}(t^3)$ and can follow the same way as the one we have used to prove the linearity. □

# 4 Linear Lie groups

Although we have investigated abstract Lie groups and their Lie algebras, hereafter we mainly treat linear Lie groups. Indeed, to see Lie's third theorem, proving Theorem 4 for $G = GL(n,\mathbb{R})$ is sufficient.

We first see that the Lie algebra of $GL(n,\mathbb{R})$ is actually $M(n,\mathbb{R})$ in the sense we have defined previously. The matrix exponential map or logarithm map can be defined as a convergent power series of matrices:

$$\exp A := I+\sum_{k=1}^\infty\frac{1}{k!}A^k,\qquad \log A := \sum_{k=1}^\infty\frac{(-1)^{i-1}}{k}(A-I)^k\quad(\text{for }\|A-I\|<1).$$

What we should confirm is that matrices $X \in M(n,\mathbb{R})$ determine elements of $\mathrm{Lie}(G)$: left-invariant vector fields. Given the exponential map, the operation should be (from (2); we denote by $\tilde{X}$ the operator corresponding to matrix $X$)

$$(\tilde{X}f)(x) = (\tilde{X}f)(xe^{tX})\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(xe^{tX})\bigg|_{t=0}.$$

This operator obviously commutes with $\pi_x$ and indeed becomes a vector field from the Leibniz rule. By looking at the operation at origin, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(e^{tX})\bigg|_{t=0} = \sum_{i,j=1}^{n} (Xe^{tX})_{ij} \frac{\partial}{\partial A_{ij}} f(xe^{tX})\bigg|_{t=0} = \sum_{i,j=1}^{n} X_{ij} \frac{\partial}{\partial A_{ij}} f(I),$$

where we see $f : A \mapsto f(A)$ for $A \in GL(n,\mathbb{R})$. This is indeed the general form of elements in $T_I GL(n,\mathbb{R})$.

Finally, let us confirm that the most important feature, the Lie bracket corresponds. From the same computation as in (6), it suffices to observe that

$$f(xe^{tX}e^{tY}e^{-tX}e^{-tY}) = f(x) + t^2(\widetilde{[X,Y]}f)(x) + \mathrm{O}(t^3). \tag{7}$$

Indeed, as $e^{tX} = I + tX + \frac{1}{2}t^2X^2 + \mathrm{O}(t^3)$ in general, we have

$$e^{tX}e^{tY} = I + t(X+Y) + \frac{1}{2}t^2(X^2 + 2XY + Y^2) + \mathrm{O}(t^3),$$

and so

$$e^{tX}e^{tY}e^{-tX}e^{-tY} = I + t^2[X,Y] + \mathrm{O}(t^3) = e^{t^2[X,Y]+\mathrm{O}(t^3)}.$$

Then, (7) immediately follows. Therefore, the notion of abstract Lie algebra is compatible to the linear Lie algebra $M(n,\mathbb{R})$ equipped with a commutator for the Lie group $GL(n,\mathbb{R})$.

We shall prove the linear Lie group version of Theorem 4. Recall the statement:

> Let $G$ be a Lie group. For any Lie subalgebra $\mathfrak{h}$ of $\mathrm{Lie}(G)$, there is a unique connected Lie subgroup $H \subset G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$.

The $H$ in this statement is called an *analytic subgroup* of $G$. For a Lie algebra $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ is called its Lie subalgebra if $[\mathfrak{h},\mathfrak{h}] := \{[x,y] \mid x,y \in \mathfrak{h}\}$ is included in $\mathfrak{h}$, i.e., $\mathfrak{h}$ is closed under the Lie bracket. Also, for a Lie group $G$, we call $H \subset G$ a Lie subgroup if $H$ is a subgroup of $G$ that has the Lie group structure making the inclusion $H \to G$ an immersion (Duistermaat & Kolk 2000).

Before proving the theorem, given a Lie subalgebra $\mathfrak{h} \subset M(n,\mathbb{R})$, we introduce a bounded linear operator $\mathrm{ad} : \mathfrak{h} \to \mathfrak{h}$ for each $Z \in \mathfrak{h}$ by $\mathrm{ad}(Z)W = [Z,W]$ (ad itself is indeed a bounded linear operator). As we have

$$\|\mathrm{ad}(Z)W\| = \|ZW - WZ\| \le 2\|Z\|\|W\|,$$

The operator norm of $\mathrm{ad}(Z)$ is bounded by $2\|Z\|$. We can also define $f(\mathrm{ad}(Z))$ for an analytic function $f$ on $\mathbb{R}$.

*Proof of Theorem 4 for $G = GL(n,\mathbb{R})$.* We are given a Lie subalgebra $\mathfrak{h}$ of $M(n,\mathbb{R})$. As $\mathfrak{h}$ is a finite-dimensional subspace of $M(n,\mathbb{R})$ as a vector space, Let us define $H$ by

$$H := \{e^{X_1} \cdots e^{X_m} \mid m \ge 0,\ X_1, \dots, X_n \in \mathfrak{h}\}.$$

Then $H$ is clearly a subgroup of $G$.

Let us give $H$ a manifold structure. Take a basis $\{X_1, \dots, X_m\}$ of $\mathfrak{h}$. As $\mathfrak{h} \subset M(n,\mathbb{R})$ holds, we can take $\{X_{m+1}, \dots, X_{n^2}\}$ such that $\{X_1, \dots, X_{n^2}\}$ becomes a basis of $M(n,\mathbb{R})$. Then, for a sufficiently small $\varepsilon > 0$, the map

$$\varphi_G^A : (-\varepsilon,\varepsilon)^{n^2} \to G; \quad (t_1, \dots, t_{n^2}) \mapsto A\exp(t_1X_1 + \cdots + t_{n^2}X_{n^2})$$

is a diffeomorphism onto a negihborhood of $A$ for each $A \in GL(n, \mathbb{R})$ and the family of (inverse of) these maps makes an analytic atlas of $GL(n, \mathbb{R})$ (a special case of Theorem 10). Then, define the manifold structure of $H$ by the restriction of this map, i.e.,

$$\varphi_H^A : (-\varepsilon, \varepsilon)^m \to H; \quad (t_1, \dots, t_m) \mapsto A \exp(t_1 X_1 + \cdots + t_m X_m).$$

Then, $\{(\varphi^A)^{-1}\}_{A \in H}$ makes an analytic atlas of $H$. Indeed, $(\varphi_G^A)^{-1} \circ \varphi_H^B$ is analytic and injective, and coincides with $(\varphi_H^A)^{-1} \circ \varphi_H^B$ on $(\varphi_H^B)^{-1}(\operatorname{Im} \varphi_H^A \cap \operatorname{Im} \varphi_H^B)$. As the latter has an explicit analytic inverse, $H$ is now an analytic manifold. The inclusion map $H \to G$ is then clearly an immersion.

We finally prove that $H$ is a Lie group. As $\{Ae^X \mid X \in \mathfrak{h},\ \|X\| < \delta\}$ defines a neighborhood system of $A \in H$, it suffices to prove for $A, B \in H$ that

$$(\mathfrak{h}, \mathfrak{h}) \to H; \quad (X, Y) \mapsto (Ae^X)^{-1} B e^Y$$

is analytic at $(X, Y) = 0$ (it is equivalent to the analyticity of both of the group operations). Then, it suffices to prove that there is a $\delta > 0$ such that $\|X\|, \|Y\| < \delta \Rightarrow (Ae^X)^{-1} B e^Y \in \operatorname{Im} \varphi_H^{A^{-1}B}$, because the desired analyticity then follows from the analyticity of $G$.

Let us write $A^{-1}B = e^{Z_1} \cdots e^{Z_k}$ by some $Z_1, \dots, Z_k$. Then, we have

$$(Ae^X)^{-1} B e^Y = e^X A B e^Y = e^X e^{Z_1} \cdots e^{Z_k} e^Y. \tag{8}$$

In general, for any $W, Z \in \mathfrak{h}$ and $t \in \mathbb{R}$ we have

$$e^{-Z} e^W e^Z = e^{-Z} \left( \sum_{i=0}^{\infty} \frac{1}{i!} \right) W^i e^Z = \sum_{i=0}^{\infty} \frac{1}{i!} (e^{-Z} W e^Z)^i = \exp(e^{-Z} W e^Z).$$

Here, $e^{-tZ} W e^{tZ}$ is analytic for all $t \in \mathbb{R}$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-tZ} W e^{tZ}) = e^{-tZ}(-Z) W e^{tZ} + e^{-tZ} W Z e^{tZ} = e^{-tZ} \operatorname{ad}(-Z) W e^{tZ}.$$

Hence, we inductively obtain

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}(e^{-tZ} W e^{tZ}) = e^{-tZ} \operatorname{ad}(-Z)^i W e^{tZ} \qquad \therefore\ e^{-Z} W e^Z = e^{\operatorname{ad}(-Z)} W$$

As $\mathfrak{h}$ is a closed subspace of $M(n, \mathbb{R})$ and also closed under the Lie bracket, we have $e^{-Z} W e^Z \in \mathfrak{h}$. Therefore, we have a sequence $W_0, W_1, \dots, W_k \in \mathfrak{h}$ with $W_0 = X$ such that

$$e^{W_{j-1}} e^{Z_j} = e^{Z_j} e^{W_j}, \qquad j = 1, \dots, k.$$

Therefore, we have $(Ae^X)^{-1} B e^Y = A B e^{W_k} e^Y$. Note that $W_k \to 0$ as $X \to 0$. Hence, it suffices to prove that there exists a $\delta > 0$ such that

$$X, Y \in \mathfrak{h},\ \|X\|, \|Y\| \le \delta \Longrightarrow \log(e^X e^Y) \in \mathfrak{h}.$$

To prove this, define $Z(t) := \log(e^{tX} e^Y)$. Note that

$$\|e^{tX} e^Y - I\| \le \|e^{tX}(e^Y - I)\| + \|e^{tX} - I\| \le e^{\|tX\|}(e^{\|Y\|} - 1) + e^{\|tX\|} - 1$$

holds, so $Z(t)$ is analytic on $(-2, 2)$ if $\|X\|$ and $\|Y\|$ are sufficiently small. For such $X$ and $Y$, as $Z(t+s) = Z(t) + sZ'(t) + \mathrm{o}(s)$ holds for $t \in [0, 1]$, we have

$$\left. \frac{\mathrm{d}}{\mathrm{d}s} e^{Z(t) + sZ'(t)} \right|_{s=0} = \left. \frac{\mathrm{d}}{\mathrm{d}t} e^{Z(t)} \right|_{s=0} = X e^{Z(t)}. \tag{9}$$

Additionally, there is a fomula for matrices $P, Q \in M(n, \mathbb{R})$: (Kobayashi & Oshima 2005, Theorem 5.54)

$$\left. \frac{\mathrm{d}}{\mathrm{d}s} e^{P + sQ} \right|_{s=0} = e^P f(\operatorname{ad}(P)) Q,$$

where $f(x) = \frac{1-e^{-x}}{x}$ is an analytic function. As $f(0) = 1$, $g := 1/f$ is analytic in a certain interval. Then, if $\|Z(t)\|$ is sufficiently small (which can be achieved by sufficiently small $\|X\|$ and $\|Y\|$), we can apply it to (9) and obtain

$$Z'(t) = g(\mathrm{ad}(Z(t)))e^{-Z(t)}Xe^{Z(t)} = g(\mathrm{ad}(Z(t)))e^{\mathrm{ad}(-Z(t))}X.$$

As both sides are analytic on $t \in (-2, 2)$ if $\|X\|$ and $\|Y\|$ are sufficiently small, we can inductively know $Z^{(\ell)}(0)$ by comparing the coefficients of $t^{\ell-1}$ for $\ell = 1, 2, \ldots$. At the same time, we see that $Z(0) = Y$ and $Z^{(\ell)}(0)$ is generated by $X, Z(0), \ldots, Z^{\ell-1}(0)$ and Lie brackets, and so $Z^{(\ell)}(0) \in \mathfrak{h}$. Therefore, finally we have $\log(e^X e^Y) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} Z^{(\ell)}(0) \in \mathfrak{h}$.

The fact $H$ is connected is clear from the definition. We do not prove the uniquness here. □

We finally remark that, as one can see from the argument in the previous proof, we can explicitly calculate $\log(e^X e^Y)$ for sufficiently small $(X, Y)$ and that is not limited to linear Lie algebras but holds universally. The formula is called the Baker–Campbell–Hausdorff formula and gives

$$\log(e^X e^Y) = X + Y + [X, Y] + \frac{1}{2}[X - Y, [X, Y]] + \cdots .$$

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