The space $H^{1/2}(\mathbb{R}^n)$

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In this report, we investigate the space $H^{1/2}(\mathbb{R}^n)$. First of all, we followed largely the arguments in Lieb & Loss (2001), but the flow and proofs are largely different from those in the book. I have tried to prove most of the statements by myself to deepen my understanding.

In Section 1, we describe the physical background of the space a little, define the space $H^{1/2}(\mathbb{R}^n)$, and explain that the definition is natural to the motivation. In Section 2, we introduce the Poisson kernel and describe its connection to $H^{1/2}(\mathbb{R}^n)$. As the statement $f \in H^{1/2}(\mathbb{R}^n) \Rightarrow |f| \in H^{1/2}(\mathbb{R}^n)$ is simple but interesting, we provide a general argument behind this. In Section 3, we prove the density of $C_c(\mathbb{R}^n)$ in $H^{1/2}(\mathbb{R}^n)$. To do so, we introduce the Schwartz space $S(\mathbb{R}^n)$ and observe the densely embedded sequence $C_c(\mathbb{R}^n) \hookrightarrow S(\mathbb{R}^n) \hookrightarrow H^{1/2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$, though the last inclusion $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is proven in Section 1. In the final section, we shall see that we also have a continuous embedding $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ in a certain range of $p$ (Sobolev inequality).

Throughout this report, let $L^2(\mathbb{R}^n)$ denote the $\mathbb{C}$-valued Lebesgue space and let $\langle f, g \rangle$ be the inner product defined as

$$\langle f, g \rangle := \int_{\mathbb{R}^n} \overline{f(x)}g(x)\,dx$$

for $f, g \in L^2(\mathbb{R}^n)$, where $\overline{z}$ denotes the complex conjugate of $z$ in general. We also let $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) ; f \mapsto \mathcal{F}[f]$ be the usual Fourier transform defined as the isometric extension of the map

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \omega \cdot x}\,dx, \quad \omega \in \mathbb{R}^n$$

defined on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

1 Motivation and definition

We first start with giving motivation to considering the fractional Sobolev space $H^{1/2}(\mathbb{R}^n)$. In the physical background, $n$ should be regarded as 3 in the description given below, but we keep using the notation of $\mathbb{R}^n$, as it finally connects to the general $H^{1/2}(\mathbb{R}^n)$.

According to Greiner (2000, Chapter 1), the following second-order wave equation called the *Klein-Gordon equation* is important in relativistic quantum mechanics:

$$\left(\frac{\partial^2}{c^2\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)\psi = 0,$$
where $\psi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}; (t,x) \mapsto \psi(t,x)$ is the wave function, $c$ is the speed of light, $\hbar$ is the Dirac constant, and $m$ denotes the mass of the free particle we consider. Note also that $\Delta$ represents the Laplacian with respect to only $x$. In this regime, the energy operator is formally written as

$$E := \sqrt{-(\hbar c)^2 \Delta + (mc^2)^2}.$$  

We shall mathematically define this operator on an appropriate subset of $L^2(\mathbb{R}^n)$ by using the Fourier transform. As

$$(-\hbar c)^2 \Delta + (mc^2)^2 \psi = \mathcal{F}^{-1} \left[ \left((2\pi\hbar c)^2 |\omega|^2 + (mc^2)^2 \right) \mathcal{F}[\psi] \right]$$

holds by the usual identity of Fourier transforms (on an appropriate subset of $L^2(\mathbb{R}^n)$), we should define

$$E \psi = \sqrt{-(\hbar c)^2 \Delta + (mc^2)^2} \psi := \mathcal{F}^{-1} \left[ \sqrt{(2\pi\hbar c)^2 |\omega|^2 + (mc^2)^2} \mathcal{F}[\psi] \right].$$

However, there remains a problem: on which subset of $L^2(\mathbb{R}^n)$, does this operator $E$ becomes well-defined ($E \psi \in L^2(\mathbb{R}^n)$) and can we compute the “expectation” of it? The latter requirement comes from the fact that in quantum mechanics the expectation of a physical quantity $E$ we observe is given by

$$\int_{\mathbb{R}^n} \overline{\psi(x)}(E \psi)(x) \, dx.$$  

The latter condition is indeed stronger, and so our requirement can be shown to be equivalent with

$$\int_{\mathbb{R}^n} \sqrt{(2\pi\hbar c)^2 |\omega|^2 + (mc^2)^2} |\mathcal{F}[\psi]|^2 \, d\omega.$$  

by using the isometric property of Fourier transform. Since we clearly have the order evaluation (we show formally in the proof of Proposition 3)

$$\sqrt{(2\pi\hbar c)^2 |\omega|^2 + (mc^2)^2} = \Theta(1 + |\omega|),$$

the space $H^{1/2}(\mathbb{R}^n)$ defined below is what we want (though there is no topological necessity for the coefficient $2\pi$ of $|\omega|$, we follow the definition of Lieb & Loss (2001)).

**Definition 1.** Define the fractional Sobolev space $H^{1/2}(\mathbb{R}^n)$ as the set of all functions $f \in L^2(\mathbb{R}^n)$ satisfying

$$\| f \|_{H^{1/2}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + 2\pi |\omega|) |\mathcal{F}[f](\omega)|^2 \, d\omega < \infty.$$  

This norm $\| \cdot \|_{H^{1/2}(\mathbb{R}^n)}$ naturally induces an inner product.

**Theorem 2.** $H^{1/2}(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^{1/2}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \overline{\mathcal{F}[f](\omega)} \mathcal{F}[g](\omega)(1 + 2\pi |\omega|) \, d\omega.$$  

Moreover, $H^{1/2}(\mathbb{R}^n)$ is continuously embedded in $L^2(\mathbb{R}^n)$.

**Proof.** Note that if $f, g \in H^{1/2}(\mathbb{R}^n)$, then $f + g \in H^{1/2}(\mathbb{R}^n)$ also holds. This follows from

$$|\mathcal{F}[f] + \mathcal{F}[g]|^2 \leq 2|\mathcal{F}[f]|^2 + |\mathcal{F}[g]|^2.$$  

Hence $H^{1/2}(\mathbb{R}^n)$ is a linear subspace of $L^2(\mathbb{R}^n)$ and $\langle \cdot, \cdot \rangle_{H^{1/2}(\mathbb{R}^n)}$ is clearly an inner product on it. Let us prove that this is indeed a Hilbert space. Let $\{f_n\}_{n \geq 1}$ be a Cauchy sequence in terms of this inner product. As we have

$$\| f_n - f_m \|^2_{H^{1/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\mathcal{F}[f_n](\omega) - \mathcal{F}[f_m](\omega)|^2 (1 + 2\pi |\omega|) \, d\omega,$$  

we observe...
it is equivalent to \( \{ \mathcal{F}[f_n] \}_{n \geq 1} \) being a Cauchy sequence in the \( L^2 \)-space \( L^2(\mathbb{R}^n, (1 + 2\pi|\omega|) \, d\omega) \). From the completeness of the \( L^2 \)-space, there exists a \( g \in L^2(\mathbb{R}^n, (1 + 2\pi|\omega|) \, d\omega) \) such that \( \mathcal{F}[f_n] \to g \) in \( L^2(\mathbb{R}^n, (1 + 2\pi|\omega|) \, d\omega) \). Therefore, \( \mathcal{F}^{-1}[g] \) is the limit of \( f_n \) in \( H^{1/2}(\mathbb{R}^n) \).

Also, for an \( f \in H^{1/2}(\mathbb{R}^n) \), as

\[
\|f\|_{H^{1/2}(\mathbb{R}^n)}^2 = \|\mathcal{F}[f]\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} 2\pi|\omega| |\mathcal{F}[f](\omega)|^2 \, d\omega \geq \|f\|_{L^2(\mathbb{R}^n)}^2
\]

by the isometry of the Fourier transform, the inclusion map \( H^{1/2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is continuous.

Going back to the physical background, we can formally prove the following equivalence.

**Proposition 3.** For arbitrary \( f \in L^2(\mathbb{R}^n) \) and \( m > 0 \), the integration (2) is finite if and only if \( f \in H^{1/2}(\mathbb{R}^n) \).

**Proof.** It suffices to prove there exist universal positive constants \( C_0 \) and \( C_1 \) such that

\[
C_0(1 + 2\pi|\omega|) \leq \sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} \leq C_1(1 + 2\pi|\omega|).
\]

The existence of \( C_0 \) follows from the AM-GM inequality:

\[
\sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} \geq \sqrt{\frac{(mc^2 + 2\pi\hbar c|\omega|)^2}{2}} \geq \min \left\{ \frac{mc^2}{\sqrt{2}}, \frac{\hbar c}{\sqrt{2}} \right\} (1 + 2\pi|\omega|).
\]

The other direction can also be shown as follows:

\[
\sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} \leq mc^2 + 2\pi\hbar c|\omega| \leq \max \{mc^2, \hbar c\} (1 + 2\pi|\omega|).
\]

So the proof is complete.

\[\square\]

## 2 Characterization via Poisson kernel

Although we have established the space \( H^{1/2}(\mathbb{R}^n) \), it is still difficult to handle as it is only discussed in terms of the Fourier transformation.

In this section, we shall see the operator \( \sqrt{-\Delta} \) as the limit of more tractable operators of the form \( \frac{1}{t} \left( 1 - e^{-t\sqrt{-\Delta}} \right) \) for \( t > 0 \). More formally, we define the following:

**Definition 4.** For each \( t > 0 \), define the Poisson kernel \( P_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \) as

\[
P_t(x, y) := \int_{\mathbb{R}^n} \exp \left( -2\pi t|\omega| + 2\pi i \omega^\top (x - y) \right) \, d\omega.
\]

Remark that this kernel act (as an operator) on functions \( f : \mathbb{R}^n \to \mathbb{C} \) as

\[
P_t f(x) := \int_{\mathbb{R}^n} P_t(x, y) f(y) \, dy.
\]

If we write \( \varphi_t(z) := P_t(y + z, y) \), then \( P_t f = \varphi_t * f \) is the definition, where \( * \) denotes the convolution. Here, \( \varphi_t \) is (defined as) the inverse Fourier transform of \( e^{-2\pi t|\omega|} \), and so we have, from the relation between the Fourier transform and the convolution, that

\[
P_t f = \mathcal{F}^{-1} \mathcal{F} [\varphi_t * f] = \mathcal{F}^{-1} \left[ e^{-2\pi t|\omega|} \mathcal{F}[f] \right].
\]

This in particular implies that \( P_t f \) is well-defined if \( f \in L^2(\mathbb{R}^n) \). Note also that \( P_t \) can be regarded as \( e^{-t\sqrt{-\Delta}} \) because \( \sqrt{-\Delta} \) is defined as \( \sqrt{-\Delta} f = \mathcal{F}^{-1} [2\pi|\omega| \mathcal{F}[f]] \) similarly as we have defined \( E \). Then, we obtain the following assertion, which characterizes \( H^{1/2}(\mathbb{R}^n) \) in a different way.
Theorem 5. A function $f \in L^2(\mathbb{R}^n)$ is contained in $H^{1/2}(\mathbb{R}^n)$ if and only if

$$I_t(f) := \frac{1}{t} ((f,f) - (f,P_t f))$$

is bounded on $t > 0$, i.e., $\sup_{t>0} I_t(f) < \infty$ holds. If $\sup_{t>0} I_t(f) < \infty$ holds, then

$$\sup_{t>0} I_t(f) = \lim_{t \searrow 0} I_t(f) = (f,\sqrt{-\Delta} f)$$

also holds.

Proof. By using the isometric property of Fourier transformation, we can rewrite $I_t(f)$ as

$$I_t(f) = \frac{1}{t} ((\mathcal{F}[f],\mathcal{F}[f]) - (\mathcal{F}[f],\mathcal{F}[P_t f]))$$

$$= \int_{\mathbb{R}^n} \frac{1-e^{-2\pi t|\omega|}}{t} |\mathcal{F}[f](\omega)|^2 \, d\omega. \quad (3)$$

Let $g(t) := t^{-1}(1-e^{-t})$ on $t > 0$. Then,

$$\frac{d}{dt}g(t) = \frac{e^{-t}t - (1-e^{-t})}{t^2} = \frac{1+t-e^t}{t^2e^t} \leq 0$$

holds since $e^t \geq 1+t$. Therefore, $g$ is monotone decreasing on $t > 0$ and satisfies $\lim_{t \searrow 0} g(t) = 1$. In terms of $g$, we obtain, from (3), the monotone convergence

$$I_t(f) = \int_{\mathbb{R}^n} g(t) 2\pi |\omega| |\mathcal{F}[f](\omega)|^2 \, d\omega \to \int_{\mathbb{R}^n} 2\pi |\omega| |\mathcal{F}[f](\omega)|^2 \, d\omega \quad (t \searrow 0),$$

where we have used the monotone convergence theorem for the limit. This immediately implies the latter part of the statement. For the former part, as $\mathcal{F}[f] \in L^2(\mathbb{R}^d)$, $f \in H^{1/2}(\mathbb{R}^d)$ is equivalent to $\lim_{t \searrow 0} I_t(f) < \infty$ (from the above limit). From the monotonicity of $I_t(f)$ with respect to $t$, we can conclude that this is equivalent to $\sup_{t>0} I_t(f) < \infty$. \qed

We next prove an interesting result: for each $f \in H^{1/2}(\mathbb{R}^n)$, $|f| \in H^{1/2}(\mathbb{R}^n)$ also holds. We shall prove this assertion from a little broader perspective.

Proposition 6. Let $k : \mathbb{R}^d \to \mathbb{C}$ be an integrable function such that $k(-x) = k(x)$ holds for any $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} k(x) \, dx = 1$ holds. Then, for an arbitrary $f \in L^2(\mathbb{R}^d)$,

$$(f,f) - (f,k*f) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} k(x-y)|f(x) - f(y)|^2 \, dx \, dy$$

holds.

Proof. From Young’s inequality (e.g., Bogachev 2007, Theorem 3.9.4), we know $k*f$ is defined almost everywhere and in $L^2(\mathbb{R}^d)$. This is also true for $|k|*|f|$, so we can use Fubini’s theorem to obtain

$$\langle f, k*f \rangle = \int_{\mathbb{R}^d} \overline{f(x)}(k*f)(x) \, dx$$

$$= \int_{\mathbb{R}^d} \overline{f(x)} \int_{\mathbb{R}^d} k(x-y)f(y) \, dy \, dx$$

$$= \iint_{\mathbb{R}^d \times \mathbb{R}^d} k(x-y)f(x)f(y) \, dx \, dy. \quad (4)$$

From $k(x-y) = k(y-x)$, we also have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} k(x-y)f(x)f(y) \, dx \, dy = \langle f, k*f \rangle. \quad (5)$$
From the assumption \( \int_{\mathbb{R}^d} k(x) \, dx = 1 \), we have
\[
\int_{\mathbb{R}^d} k(x - y) \, dx = 1, \quad \int_{\mathbb{R}^d} k(x - y) \, dy = \int_{\mathbb{R}^d} k(y - x) \, dy = 1,
\]
and so
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} k(x - y)|f(x)|^2 \, dx \, dy = \iint_{\mathbb{R}^d \times \mathbb{R}^d} k(x - y)|f(y)|^2 \, dx \, dy = 1 \tag{6}
\]
holds.

By combining (4), (5), (6), we finally obtain the desired equality. \( \Box \)

If \( k \) is additionally nonnegative real in the previous proposition, we have the following assertion.

**Proposition 7.** Let \( k : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \) be an even function with \( \int_{\mathbb{R}^d} k(x) \, dx = 1 \). Then, for an arbitrary \( f \in L^2(\mathbb{R}^d) \),
\[
\langle |f|, |f| \rangle - \langle |f|, k \ast |f| \rangle \leq \langle f, f \rangle - \langle f, k \ast f \rangle.
\]
holds.

**Proof.** From Proposition 6 and the assumption \( k \geq 0 \), it suffices to prove
\[
|f(x) - f(y)| \geq \| |f(x)| - |f(y)| \|
\]
but this just a triangle inequality, so the proof is complete. \( \Box \)

From this proposition and the first part of Theorem 5, the result \( f \in H^{1/2}(\mathbb{R}^n) \Rightarrow f \in H^{1/2}(\mathbb{R}^n) \)
follows if we prove that each \( \varphi_t \) (such that \( \varphi_t(x - y) = P_t(x, y) \)) satisfies the condition. The integrability and nonegativity follows as we can write \( \varphi_t \) explicitly as follows (Stein & Weiss 1971, Theorem 1.14):
\[
\varphi_t(x) = \Gamma \left( \frac{n + 1}{2} \right) \pi^{-\frac{n+1}{2}} \frac{t}{(t^2 + |x|^2)^\frac{n+1}{2}}.
\]
The assumption \( \int_{\mathbb{R}^d} \varphi_t(x) \, dx = 1 \) can be proven by using that this integral is indeed the value of \( \mathcal{F}[\varphi_t](0) \) and the Fourier transform is \( \mathcal{F}[\varphi_t](\omega) = e^{-\pi t|\omega|} \) by definition. Therefore, we finally obtain the following assertion.

**Theorem 8.** For an arbitrary \( f \in H^{1/2}(\mathbb{R}^n), \langle |f| \rangle \in H^{1/2}(\mathbb{R}^n) \) holds.

Remarkably, the generality of Proposition 7 yields an analogous result also for \( H^1(\mathbb{R}^n) \), where we use the heat kernel instead of the Poisson kernel.

Note also that there are parallel results for relativistic kinetic energy, i.e., the operator \( \sqrt{-\Delta} \) instead of \( \sqrt{-\Delta} \). However, in this report, we mainly consider the operator \( \sqrt{-\Delta} \) for simplicity and omit the relativistic counterpart.

### 3 Density

As is common when we define new function classes, we shall prove that the space of compactly supported smooth functions is a dense subset of \( H^{1/2}(\mathbb{R}^n) \). To do so in a different way from the approach adopted in Lieb & Loss (2001), we introduce a few properties of the Schwartz space. Although these are well-known, for example, we can find them in Grafakos (2008, Chapter 2).

For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \), we write
\[
|\alpha| := \alpha_1 + \cdots + \alpha_n, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial^\alpha := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}
\]
for each \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).
Definition 9. A smooth function \( f : \mathbb{R}^n \to \mathbb{C} \) is called a Schwartz function (or a rapidly decreasing function) if it satisfies

\[
\rho_{\alpha, \beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty
\]

for all multi-indices \( \alpha, \beta \in \mathbb{Z}_{\geq 0}^n \). We denote the space of all Schwartz functions by \( \mathcal{S}(\mathbb{R}^n) \).

\( \mathcal{S}(\mathbb{R}^n) \) can be metrized by the sequence of seminorms \( \rho_{\alpha, \beta} \). For example, we can define a metric

\[
d_{\mathcal{S}}(f, g) := \sum_{\alpha, \beta \in \mathbb{Z}_{\geq 0}^n} \min\{1, \rho_{\alpha, \beta}(f - g)\}.
\]

Here \( d_{\mathcal{S}}(f, g) \leq 1 \) always holds. \( (\mathcal{S}(\mathbb{R}^n), d_{\mathcal{S}}) \) can be shown to be a complete metric space.

For this Schwartz space \( \mathcal{S}(\mathbb{R}^n) \), the following theorem shows the good compatibility of the Schwartz space and the Fourier transform.

Theorem 10. The image of the Fourier transform \( \mathcal{F} \) on \( \mathcal{S}(\mathbb{R}^n) \) coincides with \( \mathcal{S}(\mathbb{R}^n) \), and the reduced map \( \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is a homeomorphism with respect to the metric \( d_{\mathcal{S}} \).

Proof. As an elementary properties of the Fourier transform, for \( f \in \mathcal{S}(\mathbb{R}^n) \), we have \( \mathcal{F}[\partial^\alpha f](\omega) = (2\pi i \omega)^\alpha \mathcal{F}[f](\omega) \) for each multi-index \( \alpha \). From the fact that \( \mathcal{F}^{-1}[g](x) = \mathcal{F}[g](−x) \) holds for each \( g \in \mathcal{S}(\mathbb{R}^n) \), we also have

\[
\mathcal{F}^{-1}[\partial^\alpha g](x) = \mathcal{F}[\partial^\alpha g](−x) = (−2\pi i)^\alpha \mathcal{F}[g](−x) = (−2\pi i)^\alpha \mathcal{F}^{-1}[g](x)
\]

for an \( \alpha \in \mathbb{Z}_{\geq 0}^n \), and this implies \( \partial^\alpha \mathcal{F}[f] = \mathcal{F}[(−2\pi i)^\alpha f] \).

Therefore, we have, for each multi-indices \( \alpha \) and \( \beta \),

\[
\rho_{\alpha, \beta}(\mathcal{F}[f]) = \|\omega^\alpha \partial^\beta \mathcal{F}[f](\omega)\|_{L^\infty} = \|\omega^\alpha \mathcal{F}[(−2\pi i)^\beta f]\|_{L^\infty} \leq \frac{(2\pi)^{|\beta|−|\alpha|}}{(2\pi i)^{|\alpha|}} \|\partial^\alpha x^\beta f\|_{L^1} < \infty
\]

as desired. Here, the last \( L^1 \)-integrability of \( \partial^\alpha x^\beta f \) follows as the integrand is written as a finite sum of (polynomial) \( \times \partial^\alpha f \) and this decreases faster than \( (1 + |x|^{2n})^{-1} \), for example, which is integrable.

Therefore, we have shown that the image of \( \mathcal{F} \) on \( \mathcal{S}(\mathbb{R}^n) \) is included in \( \mathcal{S}(\mathbb{R}^n) \). As the Fourier inversion \( \mathcal{F}^{-1} \) has almost the same expression as \( \mathcal{F} \), we can similarly prove that this map is a self-bijection.

For the continuity with respect to \( d_{\mathcal{S}} \), it suffices to prove that \( \rho_{\alpha, \beta}(\mathcal{F}[f]) \to 0 \) as \( d_{\mathcal{S}}(f, 0) \to 0 \) for each \( \alpha, \beta \) as \( d_{\mathcal{S}} \) is translation-invariant (the case of \( \mathcal{F}^{-1} \) can also be done from this via variable transformation). We can prove this by refining the argument of proving \( \rho_{\alpha, \beta}(\mathcal{F}[f]) < \infty \). Indeed, by writing \( \partial^\alpha x^\beta f = \sum_{(\gamma, \delta) \in \Gamma} c_{\gamma, \delta} x^\gamma \partial^\delta f \), where \( \Gamma = \Gamma(\alpha, \beta) \) is a finite set of pair of multi-indices and \( c_{\gamma, \delta} \) are constants, we have

\[
(1 + |x|^{2n}) \partial^\alpha x^\beta f = \sum_{(\gamma, \delta) \in \Gamma} c_{\gamma, \delta} (1 + |x|^{2n}) x^\gamma \partial^\delta f = \sum_{(\gamma, \delta) \in \Gamma} c_{\gamma, \delta} \sum_{\epsilon \in E} c'_\epsilon x^{\gamma + \epsilon} \partial^\delta f,
\]

where \( E \) is a finite set of multi-indices such that \( 1 + |x|^{2n} = \sum_{\epsilon \in E} c'_\epsilon x^\epsilon \). Therefore,

\[
\sup_{x \in \mathbb{R}^n} (1 + |x|^{2n}) |\partial^\alpha x^\beta f| \leq \sum_{(\gamma, \delta, \epsilon) \in \Gamma \times E} c_{\gamma, \delta} c'_\epsilon \rho_{\gamma + \epsilon, \delta}(f)
\]

holds, and so we finally obtain

\[
\rho_{\alpha, \beta}(\mathcal{F}[f]) \leq (2\pi)^{|\beta|−|\alpha|} \|\partial^\alpha x^\beta f\|_{L^1} \leq \left( \int_{\mathbb{R}^n} \frac{dx}{1 + |x|^{2n}} \right) \sum_{(\gamma, \delta, \epsilon) \in \Gamma \times E} c_{\gamma, \delta} c'_\epsilon \rho_{\gamma + \epsilon, \delta}(f) \to 0
\]

as \( d_{\mathcal{S}}(f, 0) \) tends to zero. \( \square \)
We next prove the relation of $S(\mathbb{R}^n)$ with other common function spaces. Denote the space of all smooth (i.e., infinitely differentiable) functions with a compact support by $C_c(\mathbb{R}^n)$. $C_c(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ is obvious.

**Theorem 11.** $C_c(\mathbb{R}^n)$ is a dense subset of $S(\mathbb{R}^n)$.

**Proof.** Take an arbitrary $f \in S(\mathbb{R}^n)$. It suffices to prove that for each positive integer $m$ and $\varepsilon > 0$, there exists a $g \in C_c(\mathbb{R}^n)$ such that $\rho_{\alpha,\beta}(f - g) < \varepsilon$ for all $|\alpha|, |\beta| \leq m$.

Take a function $h \in C_c(\mathbb{R}^n)$ such that $h(x) = 1$ on $|x| \leq 1$, $h(x) \in [0,1]$ on $|x| \in [1,2]$, and $h(x) = 0$ on $|x| \geq 2$. Such a function indeed exists; it can be constructed by exploiting the one-dimensional smooth function

$$t \mapsto \begin{cases} 0 & (t \leq 0) \\ e^{-1/t} & (t > 0) \end{cases},$$

but we omit the details here. For a positive integer $N$, define $h_N(x) := h(N^{-1}x)$. Then, for each multi-index $\alpha$, $\partial^{\alpha}h_N(x) = N^{-\alpha}(\partial^{\alpha}h)(N^{-1}x)$ holds. In particular, we have

$$N \max_{0<|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^{\alpha}h_N(x)| \leq \max_{0<|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^{\alpha}h(x)| =: C < \infty$$

for each $N$. We shall prove that $\lim_{N \to \infty} \rho_{\alpha,\beta}(h_N f - f) = 0$ for all $|\alpha|, |\beta| \leq m$. Fix $\alpha$ and $\beta$ such that $|\alpha|, |\beta| \leq m$. Then, from the Leibniz rule, we have

$$x^{\alpha} \partial^{\beta}(h_N f - f) = (h_N - 1)x^{\alpha} \partial^{\beta} f + \sum_{\beta_1 + \beta_2 = \beta, \beta_1 \neq 0} c_{\beta_1,\beta_2}(\partial^{\beta_1}h_N)(x^{\alpha} \partial^{\beta_2} f)$$

for some positive integer constants $c_{\beta_1,\beta_2}$. By using (7), we obtain

$$\sup_{x \in \mathbb{R}^n} |x^{\alpha} \partial^{\beta}(h_N f - f)| \leq \sup_{|x| \geq N} |x^{\alpha} \partial^{\beta} f(x)| + \frac{C}{N} \sum_{\beta_1 + \beta_2 = \beta, \beta_1 \neq 0} c_{\beta_1,\beta_2} \rho_{\alpha,\beta_2}(f).$$

The second term in the right-hand side obviously tends to zero, whereas the convergence first term is also clear from $\sup_{x \in \mathbb{R}^n} |x^{\alpha} \partial^{\beta} f(x)| < \infty$. Hence, the assertion of the theorem holds.

From the usual density result of $C_c(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ (e.g., Theorem 1.5.8 in the lecture note), we can prove the following theorem.

**Theorem 12.** For $1 \leq p < \infty$, $S(\mathbb{R}^n)$ is a dense subset of $L^p(\mathbb{R}^n)$. Moreover, the inclusion map $S(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous.

**Proof.** As $C_c(\mathbb{R}^n)$ is a dense subset of $L^p(\mathbb{R}^n)$, the former assertion is clear just from the inclusion $C_c(\mathbb{R}^n) \subset S(\mathbb{R}^n)$.

For the latter part, we follow the proof of Proposition 2.2.6 in Grafakos (2008). For an arbitrary $f \in S(\mathbb{R}^n)$, we have

$$\|f\|_{L^p} = \int_{|x| \leq 1} |f(x)|^p \, dx + \int_{|x| \geq 1} \frac{1}{|x|^{n+1}} |f(x)|^p \, dx$$

$$\leq \sup_{x \in \mathbb{R}^n} |f(x)|^p + \left( \int_{|x| \geq 1} \frac{dx}{|x|^{n+1}} \right) \sup_{x \in \mathbb{R}^n} \left| x^{\frac{n+1}{p}} f(x) \right|^p,$$

and so it follows that $f_n \to f$ in $d_S$ implies $f_n \to f$ in $L^p$. 

Let us go back to the space $H^{1/2}(\mathbb{R}^n)$.

**Theorem 13.** $S(\mathbb{R}^n)$ is a dense subset of $H^{1/2}(\mathbb{R}^n)$. Moreover, the inclusion $S(\mathbb{R}^n) \subset H^{1/2}(\mathbb{R}^n)$ is continuous.
Proof. The inclusion follows from $\mathcal{F}(S(\mathbb{R}^n)) = S(\mathbb{R}^n)$ (Theorem 10). Indeed, as $\mathcal{F}[f] \in S(\mathbb{R}^n)$ holds for each $f \in S(\mathbb{R}^n)$, $(1 + 2\pi |\omega|)|\mathcal{F}[f](\omega)|^2$ is integrable.

We next prove the density. Take an arbitrary $f \in H^{1/2}(\mathbb{R}^n)$. Then, from the definition of $H^{1/2}(\mathbb{R}^n)$, $(1 + 4\pi^2 |\omega|^2)^{1/4} \mathcal{F}[f](\omega)$ is in $L^2(\mathbb{R}^n)$:

$$
\int_{\mathbb{R}^n} (1 + 4\pi^2 |\omega|^2)^{1/2} |\mathcal{F}[f](\omega)|^2 \, d\omega \leq \int_{\mathbb{R}^n} (1 + 2\pi |\omega|)|\mathcal{F}[f](\omega)|^2 \, d\omega < \infty.
$$

Hence, from Theorem 12 with $p = 2$, there exists a sequence $g_n \in S(\mathbb{R}^n)$ which is convergent to $(1 + 4\pi^2 |\omega|^2)^{1/4} \mathcal{F}[f]$ in $L^2$. As $(1 + 4\pi^2 |\omega|^2)^{-1/4}$ is smooth, $(1 + 4\pi^2 |\omega|^2)^{-1/4} g_n$ is also smooth. We prove this function is indeed in $S(\mathbb{R}^n)$. For each multi-index $\alpha$, we can prove that

$$
\partial^\alpha (1 + 4\pi^2 |\omega|^2)^{-1/4} = \sum_{i=0}^{|\alpha|} c_i(x)(1 + 4\pi^2 |\omega|^2)^{-1/4 - i}
$$

for some polynomials $c_i$, by induction. Combining this with the Leibniz rule, we see that

$$
\sup_{x \in \mathbb{R}^n} \left| \omega^\alpha \partial^\beta \left( (1 + 4\pi^2 |\omega|^2)^{-1/4} g_n \right) \right| < \infty
$$

holds for each $\alpha$ and $\beta$. Therefore, by letting $f_n := \mathcal{F}^{-1}\left[(1 + 4\pi^2 |\omega|^2)^{-1/4} g_n\right]$, we have

$$
\|f_n - f\|_{H^{1/2}} \leq 2^{1/4} \left\| (1 + 4\pi^2 |\omega|^2)^{1/4} (\mathcal{F}[f_n] - \mathcal{F}[f]) \right\|_{L^2} = \left\| g_n - (1 + 4\pi^2 |\omega|^2)^{1/4} \mathcal{F}[f] \right\|_{L^2} \to 0
$$

as $n \to \infty$. Since we know $f_n \in S(\mathbb{R}^n)$ from Theorem 10, $S(\mathbb{R}^n)$ is dense in $H^{1/2}(\mathbb{R}^n)$.

Finally, we shall prove the continuity of the inclusion. This is done in a similar manner to the previous theorem. Indeed, we have, for $f \in S(\mathbb{R}^n)$,

$$
\|f\|^2_{H^{1/2}} \leq \int_{\mathbb{R}^n} (1 + 2\pi) |\mathcal{F}[f](\omega)|^2 \, d\omega + \int_{|\omega| \geq 1} \frac{1 + 2\pi}{|\omega|^{n+1}} |\omega|^{n+2} |\mathcal{F}[f](\omega)|^2 \, d\omega
$$

$$
\leq (1 + 2\pi) \sup_{\omega \in \mathbb{R}^n} |\mathcal{F}[f](\omega)|^2 + \left( \int_{|\omega| \geq 1} \frac{1 + 2\pi}{|\omega|^{n+1}} \, d\omega \right) \sup_{\omega \in \mathbb{R}^n} |\omega|^{n+2} |\mathcal{F}[f](\omega)|^2.
$$

Thus, as $f \mapsto \mathcal{F}[f]$ is continuous on $(S(\mathbb{R}^n), d_S)$, $S(\mathbb{R}^n) \subset H^{1/2}(\mathbb{R}^n)$ is a continuous embedding. \[\square\]

Combining those results, we finally obtain the following assertion.

**Theorem 14.** $C_c(\mathbb{R}^n)$ is a dense subset of $H^{1/2}(\mathbb{R}^n)$.

**Proof.** As $S(\mathbb{R}^n)$ is dense in $H^{1/2}(\mathbb{R}^n)$ (Theorem 13), for each $\epsilon > 0$ and $f \in H^{1/2}(\mathbb{R}^n)$, there is an $g \in S(\mathbb{R}^n)$ such that $\|g - f\|_{H^{1/2}} \leq \epsilon/2$. As $C_c(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^n)$ (Theorem 11), there exists a sequence $g_n \in C_c(\mathbb{R}^n)$ convergent to $g$ under the metric $d_S$. From the continuity of inclusion $S(\mathbb{R}^n) \subset H^{1/2}(\mathbb{R}^n)$, $\|g_n - g\|_{H^{1/2}} \to 0$ also holds. Therefore, for a sufficiently large $n$, we have

$$
\|g_n - f\|_{H^{1/2}} \leq \|g_n - g\|_{H^{1/2}} + \|g - f\|_{H^{1/2}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.
$$

As $f$ and $\epsilon$ are arbitrary, the proof is complete. \[\square\]

**4 Sobolev inequality**

In this final section, we try the generalization of the continuous embedding $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ (Theorem 2). Although the statement holds even for $p = 2n/(n - 1)$ with $n \geq 2$ (Lieb & Loss 2001, Theorem 8.4; Di Nezza et al. 2012, Theorem 6.5), we here omit the proof for that case.
We also have a counterexample. From the Hausdorff-Young inequality (Lieb & Loss 2001, Theorem 5.7; Grafakos 2008, Proposition 2.2.16), we have
\[ \|f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}\|f\|_{H^{1/2}(\mathbb{R}^n)} \]
for all \( f \in H^{1/2}(\mathbb{R}^n) \). In particular, \( H^{1/2}(\mathbb{R}^n) \) is continuously embedded in \( L^p(\mathbb{R}^n) \).

**Proof.** This is the generalization of the proof of (Lieb & Loss 2001, Theorem 8.5). The case \( p = 2 \) has already been proven in Theorem 2, where we can take \( C = 1 \). Let \( f \in H^{1/2}(\mathbb{R}^n) \). Then, for \( p > 2 \), we take \( q \in (1, 2) \) such that \( 1/p + 1/q = 1 \). Then, we have, for an \( r \) such that \( 1/r + q/2 = 1 \),
\[
\|\mathcal{F}[f]\|_{L^q}^q = \int_{\mathbb{R}^n} (1 + 2\pi|\omega|)^{-q/2} \left|\mathcal{F}[f](\omega)(1 + 2\pi|\omega|)^{1/2}\right|^q \, d\omega
\leq \left\| (1 + 2\pi|\omega|)^{-q/2} \right\|_{L^r} \left\| \mathcal{F}[f](\omega)(1 + 2\pi|\omega|)^{1/2} \right\|_{L^{2/q}}^q \quad \text{(from Hölder’s inequality)}
= \left\| (1 + 2\pi|\omega|)^{-q/2} \right\|_{L^r} \|f\|_{H^{1/2}}^q.
\]

From the Hausdorff-Young inequality (Lieb & Loss 2001, Theorem 5.7; Grafakos 2008, Proposition 2.2.16), we have \( \|f\|_{L^r} \leq \|\mathcal{F}[f]\|_{L^q} \), and so it suffices to confirm that \( (1 + 2\pi|\omega|)^{-q/2} \) is integrable.

Indeed, we have
\[
r = \frac{1}{1 - q/2} = 1 - \frac{1}{2(1-1/p)} = \frac{2(1-1/p)}{2(1-1/p) - 1} = \frac{2(p-1)}{p-2},
\]
and so
\[
\frac{qr}{2} = \frac{1}{1 - 1/p} p = \frac{p}{p-2}.
\]
As we have
\[
\int_{\mathbb{R}^n} (1 + 2\pi|\omega|)^{-p/(p-2)} \, d\omega < \infty \quad \text{(8)}
\]
when \( p/(p-2) > n \), the conclusion holds. \( \square \)

For \( n \geq 2 \), the end exponent \( p = \frac{2n}{n-1} \) is called the fractional critical exponent (Di Nezza et al. 2012). In this proof, this exponent is indeed critical in whether or not achieving (8). Then, it is natural to consider the case \( n = 1 \) and \( p = \infty \), which can be regarded as a critical case. However, there is a counterexample.

For each \( N \geq 3 \), consider the function
\[
f_N(x) = \begin{cases} \frac{1}{x \log x} & (e \leq x \leq N) \\ 0 & \text{(otherwise)} \end{cases}.
\]
Then, we have \( \|\mathcal{F}^{-1}[f_N]\|_{L^\infty} \leq \|f_N\|_{L^1} \), and this equality indeed holds as \( \mathcal{F}^{-1}[f_N] \) is continuous due to \( f_N \in L^1(\mathbb{R}) \) and
\[
\mathcal{F}^{-1}[f_N](0) = \int_{\mathbb{R}} f_N(x) \, dx = \|f_N\|_{L^1}.
\]
We can explicitly calculate as
\[
\|\mathcal{F}^{-1}[f_N]\|_{L^\infty} = \int_e^N \frac{dx}{x \log x} = \log \log N - \log \log e = \log \log N.
\]
We also have
\[
\|\mathcal{F}^{-1}[f_N]\|_{H^{1/2}}^2 \leq (1 + 2\pi) \int_e^N \frac{dx}{x^2 \log x} = (1 + 2\pi) \left( 1 - \frac{1}{\log N} \right).
\]
Therefore, we obtain
\[
\frac{\|\mathcal{F}^{-1}[f_N]\|_{L^\infty}}{\|\mathcal{F}^{-1}[f_N]\|_{H^{1/2}}} \geq (1 + 2\pi)^{-1/2} \left( 1 - \frac{1}{\log N} \right)^{1/2} \log \log N \to \infty
\]
as \( N \to \infty \). In particular, there is no Sobolev inequality like \( \| \cdot \|_{L^\infty} \leq C_{1,\infty}\| \cdot \|_{H^{1/2}} \).
References


