The space $H^{1/2}(\mathbb{R}^n)$

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In this report, we investigate the space $H^{1/2}(\mathbb{R}^n)$. First of all, we followed largely the arguments in Lieb & Loss (2001), but the flow and proofs are largely different from those in the book. I have tried to prove most of the statements by myself to deepen my understanding.

In Section 1, we describe the physical background of the space a little, define the space $H^{1/2}(\mathbb{R}^n)$, and explain that the definition is natural to the motivation. In Section 2, we introduce the Poisson kernel and describe its connection to $H^{1/2}(\mathbb{R}^n)$. As the statement $f \in H^{1/2}(\mathbb{R}^n) \Rightarrow |f| \in H^{1/2}(\mathbb{R}^n)$ is simple but interesting, we provide a general argument behind this. In Section 3, we prove the density of $C_c(\mathbb{R}^n)$ in $H^{1/2}(\mathbb{R}^n)$. To do so, we introduce the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and observe the densely embedded sequence

$$C_c(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \hookrightarrow H^{1/2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n),$$

though the last inclusion $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is proven in Section 1. In the final section, we shall see that we also have a continuous embedding $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ in a certain range of p (Sobolev inequality).

Throughout this report, let $L^2(\mathbb{R}^n)$ denote the $\mathbb{C}$-valued Lebesgue space and let $\langle f, g\rangle$ be the inner product defined as

$$\langle f, g\rangle := \int_{\mathbb{R}^n} \overline{f(x)} g(x)\,\mathrm{d}x$$

for $f, g \in L^2(\mathbb{R}^n)$, where $\overline{z}$ denotes the complex conjugate of z in general. We also let $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d); f \mapsto \mathcal{F}[f]$ be the usual Fourier transform defined as the isometric extension of the map

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \omega^\top x}\,\mathrm{d}x, \qquad \omega \in \mathbb{R}^n$$

defined on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

1 Motivation and definition

We first start with giving motivation to considering the fractional Sobolev space $H^{1/2}(\mathbb{R}^n)$. In the physical background, n should be regarded as 3 in the description given below, but we keep using the notation of $\mathbb{R}^n$, as it finally connects to the general $H^{1/2}(\mathbb{R}^n)$.

According to Greiner (2000, Chapter 1), the following second-order wave equation called the *Klein-Gordon equation* is important in relativistic quantum mechanics:

$$\left(\frac{\partial^2}{c^2 \partial t^2} - \Delta + \frac{m^2 c^2}{\hbar^2}\right)\psi = 0,$$

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where $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}; (t, x) \mapsto \psi(t, x)$ is the wave function, c is the speed of light, $\hbar$ is the Dirac constant, and m denotes the mass of the free particle we consider. Note also that Δ represents the Laplacian with respect to only x. In this regime, the energy operator is formally written as

$$E := \sqrt{-(\hbar c)^2 \Delta + (mc^2)^2}.$$

We shall mathematically define this operator on an appropriate subset of $L^2(\mathbb{R}^n)$ by using the Fourier transform. As

$$(-(\hbar c)^2 \Delta + (mc^2)2)\psi = \mathcal{F}^{-1}\left[\left((2\pi\hbar c)^2|\omega|^2 + (mc^2)^2\right)\mathcal{F}[f](\omega)\right]$$

holds by the usual identity of Fourier transforms (on an appropriate subset of $L^2(\mathbb{R}^n)$), we should define

$$E\psi = \sqrt{-(\hbar c)^2 \Delta + (mc^2)^2}\psi := \mathcal{F}^{-1}\left[\sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2}\mathcal{F}[f](\omega)\right]. \tag{1}$$

However, there remains a problem: on which subset of $L^2(\mathbb{R}^n)$, does this operator E becomes well-defined ($E\psi \in L^2(\mathbb{R}^n)$) and can we compute the "expectation" of it? The latter requirement comes from the fact that in quantum mechanics the expectation of a physical quantity E we observe is given by

$$\int_{\mathbb{R}^n} \overline{\psi(x)}(E\psi)(x)\,\mathrm{d}x.$$

The latter condition is indeed stronger, and so our requirement can be shown to be equivalent with

$$\int_{\mathbb{R}^n} \sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2}\,\left|\mathcal{F}[f](\omega)\right|^2\,\mathrm{d}\omega. \tag{2}$$

by using the isometric property of Fourier transform. Since we clearly have the order evaluation (we show formally in the proof of Proposition 3)

$$\sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} = \Theta(1 + |\omega|),$$

the space $H^{1/2}(\mathbb{R}^n)$ defined below is what we want (though there is no topological necessity for the coefficient 2π of $|\omega|$, we follow the definition of Lieb & Loss (2001)).

Definition 1. Define the fractional Sobolev space $H^{1/2}(\mathbb{R}^n)$ as the set of all functions $f \in L^2(\mathbb{R}^n)$ satisfying

$$\|f\|^2_{H^{1/2}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + 2\pi|\omega|)|\mathcal{F}[f](\omega)|^2\,\mathrm{d}\omega < \infty.$$

This norm $\|\cdot\|_{H^{1/2}(\mathbb{R}^n)}$ naturally induces an inner product.

Theorem 2. *$H^{1/2}(\mathbb{R}^n)$ is a Hilbert space with the inner product*

$$\langle f, g\rangle_{H^{1/2}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \overline{\mathcal{F}[f](\omega)}\mathcal{F}[g](\omega)(1 + 2\pi|\omega|)\,\mathrm{d}\omega.$$

Moreover, $H^{1/2}(\mathbb{R}^n)$ is continuously embedded in $L^2(\mathbb{R}^n)$.

Proof. Note that if $f, g \in H^{1/2}(\mathbb{R}^n)$, then $f + g \in H^{1/2}(\mathbb{R}^n)$ also holds. This follows from

$$|\mathcal{F}[f] + \mathcal{F}[g]|^2 \le 2|\mathcal{F}[f]|^2 + |\mathcal{F}[g]|^2.$$

Hence $H^{1/2}(\mathbb{R}^n)$ is a linear subspace of $L^2(\mathbb{R}^n)$ and $\langle\cdot,\cdot\rangle_{H^{1/2}(\mathbb{R}^n)}$ is clearly an inner product on it. Let us prove that this is indeed a Hilbert space. Let $\{f_n\}_{n\ge1}$ be a Cauchy sequence in terms of this inner product. As we have

$$\|f_n - f_m\|^2_{H^{1/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\mathcal{F}[f_n](\omega) - \mathcal{F}[f_m](\omega)|^2(1 + 2\pi|\omega|)\,\mathrm{d}\omega,$$

it is equivalent to $\{\mathcal{F}[f_n]\}_{n\geq 1}$ being a Cauchy sequence in the L^2-space $L^2(\mathbb{R}^n, (1+2\pi|\omega|)\,\mathrm{d}\omega)$. From the completeness of the L^2-space, there exists a $g \in L^2(\mathbb{R}^n, (1+2\pi|\omega|)\,\mathrm{d}\omega)$ such that $\mathcal{F}[f_n] \to g$ in $L^2(\mathbb{R}^n, (1+2\pi|\omega|)\,\mathrm{d}\omega)$. Therefore, $\mathcal{F}^{-1}[g]$ is the limit of f_n in $H^{1/2}(\mathbb{R}^n)$.

Also, for an $f \in H^{1/2}(\mathbb{R}^n)$, as

$$\|f\|_{H^{1/2}(\mathbb{R}^n)}^2 = \|\mathcal{F}[f]\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} 2\pi|\omega||\mathcal{F}[f](\omega)|^2\,\mathrm{d}\omega \geq \|f\|_{L^2(\mathbb{R}^n)}^2$$

by the isometry of the Fourier transform, the inclusion map $H^{1/2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous. □

Going back to the physical background, we can formally prove the following equivalence.

Proposition 3. *For arbitrary $f \in L^2(\mathbb{R}^n)$ and $m > 0$, the integration* (2) *is finite if and only if $f \in H^{1/2}(\mathbb{R}^n)$.*

Proof. It suffices to prove there exist universal positive constants C_0 and C_1 such that

$$C_0(1+2\pi|\omega|) \leq \sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} \leq C_1(1+2\pi|\omega|).$$

The existence of C_0 follows from the AM-GM inequality:

$$\sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} \geq \sqrt{\frac{(mc^2 + 2\pi\hbar c|\omega|)^2}{2}} \geq \min\left\{\frac{mc^2}{\sqrt{2}}, \frac{\hbar c}{\sqrt{2}}\right\}(1+2\pi|\omega|).$$

The other direction can also be shown as follows:

$$\sqrt{(2\pi\hbar c)^2|\omega|^2 + (mc^2)^2} \leq mc^2 + 2\pi\hbar c|\omega| \leq \max\left\{mc^2, \hbar c\right\}(1+2\pi|\omega|).$$

So the proof is complete. □

2 Characterization via Poisson kernel

Although we have established the space $H^{1/2}(\mathbb{R}^n)$, it is still difficult to handle as it is only discussed in terms of the Fourier transformation.

In this section, we shall see the operator $\sqrt{-\Delta}$ as the limit of more tractable operators of the form $\dfrac{1}{t}\left(1 - e^{-t\sqrt{-\Delta}}\right)$ for $t > 0$. More formally, we define the following:

Definition 4. For each $t > 0$, define the Poisson kernel $P_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ as

$$P_t(x,y) := \int_{\mathbb{R}^n} \exp\left(-2\pi t|\omega| + 2\pi i\omega^\top(x-y)\right)\mathrm{d}\omega.$$

Remark that this kernel act (as an operator) on functions $f : \mathbb{R}^n \to \mathbb{C}$ as

$$P_t f(x) := \int_{\mathbb{R}^n} P_t(x,y)f(y)\,\mathrm{d}y.$$

If we write $\varphi_t(z) := P_t(y+z, y)$, then $P_t f = \varphi_t * f$ is the definition, where $*$ denotes the convolution. Here, φ_t is (defined as) the inverse Fourier transform of $e^{-2\pi t|\omega|}$, and so we have, from the relation between the Fourier transform and the convolution, that

$$P_t f = \mathcal{F}^{-1}\mathcal{F}\left[\varphi_t * f\right] = \mathcal{F}^{-1}\left[e^{-2\pi t|\omega|}\mathcal{F}[f]\right].$$

This in particular implies that $P_t f$ is well-defined if $f \in L^2(\mathbb{R}^n)$. Note also that P_t can be regarded as $e^{-t\sqrt{-\Delta}}$ because $\sqrt{-\Delta}$ is defined as $\sqrt{-\Delta}f = \mathcal{F}^{-1}\left[2\pi|\omega|\mathcal{F}[f]\right]$ similarly as we have defined E. Then, we obtain the following assertion, which characterizes $H^{1/2}(\mathbb{R}^n)$ in a different way.

Theorem 5. *A function $f \in L^2(\mathbb{R}^n)$ is contained in $H^{1/2}(\mathbb{R}^n)$ if and only if*

$$I_t(f) := \frac{1}{t}\left(\langle f, f\rangle - \langle f, P_t f\rangle\right)$$

is bounded on $t > 0$, i.e., $\sup_{t>0} I_t(f) < \infty$ holds. If $\sup_{t>0} I_t(f) < \infty$ holds, then

$$\sup_{t>0} I_t(f) = \lim_{t\searrow 0} I_t(f) = \langle f, \sqrt{-\Delta} f\rangle$$

also holds.

Proof. By using the isometric property of Fourier transformation, we can rewrite $I_t(f)$ as

$$\begin{aligned} I_t(f) &= \frac{1}{t}\left(\langle \mathcal{F}[f], \mathcal{F}[f]\rangle - \langle \mathcal{F}[f], \mathcal{F}[P_t f]\rangle\right) \\ &= \int_{\mathbb{R}^n} \frac{1 - e^{-2\pi t|\omega|}}{t} \left|\mathcal{F}[f](\omega)\right|^2 \,\mathrm{d}\omega. \end{aligned} \tag{3}$$

Let $g(t) := t^{-1}(1 - e^{-t})$ on $t > 0$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \frac{e^{-t}t - (1 - e^{-t})}{t^2} = \frac{1 + t - e^t}{t^2 e^t} \le 0$$

holds since $e^t \ge 1 + t$. Therefore, g is monotone decreasing on $t > 0$ and satisfies $\lim_{t\searrow 0} g(t) = 1$. In terms of g, we obtain, from (3), the monotone convergence

$$I_t(f) = \int_{\mathbb{R}^n} g(t) 2\pi|\omega| \left|\mathcal{F}[f](\omega)\right|^2 \,\mathrm{d}\omega \nearrow \int_{\mathbb{R}^n} 2\pi|\omega| \left|\mathcal{F}[f](\omega)\right|^2 \,\mathrm{d}\omega \quad (t \searrow 0),$$

where we have used the monotone convergence theorem for the limit. This immediately implies the latter part of the statement. For the former part, as $\mathcal{F}[f] \in L^2(\mathbb{R}^d)$, $f \in H^{1/2}(\mathbb{R}^d)$ is equivalent to $\lim_{t\searrow} I_t(f) < \infty$ (from the above limit). From the monotonicity of $I_t(f)$ with respect to t, we can conclude that this is equivalent to $\sup_{t>0} I_t(f) < \infty$. □

We next prove an interesting result: for each $f \in H^{1/2}(\mathbb{R}^n)$, $|f| \in H^{1/2}(\mathbb{R}^n)$ also holds. We shall prove this assertion from a little broader perspective.

Proposition 6. *Let $k : \mathbb{R}^d \to \mathbb{C}$ be an integrable function such that $k(-x) = k(x)$ holds for any $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} k(x)\,\mathrm{d}x = 1$ holds. Then, for an arbitrary $f \in L^2(\mathbb{R}^d)$,*

$$\langle f, f\rangle - \langle f, k * f\rangle = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} k(x - y)|f(x) - f(y)|^2 \,\mathrm{d}x\,\mathrm{d}y$$

holds.

Proof. From Young's inequality (e.g., Bogachev 2007, Theorem 3.9.4), we know $k*f$ is defined almost everywhere and in $L^2(\mathbb{R}^d)$. This is also true for $|k| * |f|$, so we can use Fubini's theorem to obtain

$$\begin{aligned} \langle f, k * f\rangle &= \int_{\mathbb{R}^d} \overline{f(x)}(k * f)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \overline{f(x)} \int_{\mathbb{R}^d} k(x - y) f(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} k(x - y)\overline{f(x)} f(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4}$$

From $k(x - y) = k(y - x)$, we also have

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} k(x - y) f(x)\overline{f(y)}\,\mathrm{d}x\,\mathrm{d}y = \langle f, k * f\rangle. \tag{5}$$

From the assumption $\int_{\mathbb{R}^d} k(x)\,\mathrm{d}x = 1$, we have

$$\int_{\mathbb{R}^d} k(x-y)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}^d} k(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} k(y-x)\,\mathrm{d}y = 1,$$

and so

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} k(x-y)|f(x)|^2\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^d\times\mathbb{R}^d} k(x-y)|f(y)|^2\,\mathrm{d}x\,\mathrm{d}y = 1 \tag{6}$$

holds.

By combining (4), (5), (6), we finally obtain the desired equality. □

If k is additionally nonnegative real in the previous proposition, we have the following assertion.

Proposition 7. *Let $k : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be an even function with $\int_{\mathbb{R}^d} k(x)\,\mathrm{d}x = 1$. Then, for an arbitrary $f \in L^2(\mathbb{R}^d)$,*

$$\langle |f|, |f| \rangle - \langle |f|, k * |f| \rangle \leq \langle f, f \rangle - \langle f, k * f \rangle .$$

holds.

Proof. From Proposition 6 and the assumption $k \geq 0$, it suffices to prove

$$|f(x) - f(y)| \geq ||f(x)| - |f(y)|| ,$$

but this just a triangle inequality, so the proof is complete. □

From this proposition and the first part of Theorem 5, the result $f \in H^{1/2}(\mathbb{R}^n) \Rightarrow f \in H^{1/2}(\mathbb{R}^n)$ follows if we prove that each φ_t (such that $\varphi_t(x-y) = P_t(x,y)$) satisfies the condition. The integrability and nonegativity follows as we can write φ_t explicitly as follows (Stein & Weiss 1971, Theorem 1.14):

$$\varphi_t(x) = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}.$$

The assumption $\int_{\mathbb{R}^d} \varphi_t(x)\,\mathrm{d}x = 1$ can be proven by using that this integral is indeed the value of $\mathcal{F}[\varphi_t](0)$ and the Fourier transform is $\mathcal{F}[\varphi_t](\omega) = e^{-2\pi t|\omega|}$ by definition. Therefore, we finally obtain the following assertion.

Theorem 8. *For an arbitrary $f \in H^{1/2}(\mathbb{R}^n)$, $|f| \in H^{1/2}(\mathbb{R}^n)$ holds.*

Remarkably, the generality of Proposition 7 yields an analogous result also for $H^1(\mathbb{R}^n)$, where we use the heat kernel instead of the Poisson kernel.

Note also that there are parallel results for relativistic kinetic energy, i.e., the operator E instead of $\sqrt{-\Delta}$. However, in this report, we mainly consider the operator $\sqrt{-\Delta}$ for simplicity and omit the relativistic counterpart.

3 Density

As is common when we define new function classes, we shall prove that the space of compactly supported smooth functions is a dense subset of $H^{1/2}(\mathbb{R}^n)$. To do so in a different way from the approach adopted in Lieb & Loss (2001), we introduce a few properties of the Schwartz space. Although these are well-known, for example, we can find them in Grafakos (2008, Chapter 2).

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we write

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \qquad x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad \partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Definition 9. A smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is called a Schwartz function (or a rapidly decreasing function) if it satisfies

$$\rho_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial^\beta f(x)\right| < \infty$$

for all multi-indices $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We denote the space of all Schwartz functions by $\mathcal{S}(\mathbb{R}^n)$.

$\mathcal{S}(\mathbb{R}^n)$ can be metrized by the sequence of seminorms $\rho_{\alpha,\beta}$. For example, we can define a metric

$$d_{\mathcal{S}}(f, g) := \sum_{\alpha,\beta\in\mathbb{Z}^n_{\geq 0}} \frac{\min\{1, \rho_{\alpha,\beta}(f - g)\}}{2^{|\alpha|+|\beta|}}.$$

Here $d_{\mathcal{S}}(f, g) \leq 1$ always holds. $(\mathcal{S}(\mathbb{R}^n), d_{\mathcal{S}})$ can be shown to be a complete metric space.

For this Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the following theorem shows the good compatibility of the Schwartz space and the Fourier transform.

Theorem 10. *The image of the Fourier transform $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$ coincides with $\mathcal{S}(\mathbb{R}^n)$, and the reduced map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a homeomorphism with respect to the metric $d_{\mathcal{S}}$.*

Proof. As an elementary properties of the Fourier transform, for $f \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}[\partial^\alpha f](\omega) = (2\pi i\omega)^\alpha \mathcal{F}[f](\omega)$ for each multi-index α. From the fact that $\mathcal{F}^{-1}[g](x) = \mathcal{F}[g](-x)$ holds for each $g \in \mathcal{S}(\mathbb{R}^n)$, we also have

$$\mathcal{F}^{-1}[\partial^\alpha g](x) = \mathcal{F}[\partial^\alpha g](-x) = (-2\pi ix)^\alpha \mathcal{F}[g](-x) = (-2\pi ix)^\alpha \mathcal{F}^{-1}[g](x)$$

for an $\alpha \in \mathbb{Z}^n_{\geq 0}$, and this implies $\partial^\alpha \mathcal{F}[f] = \mathcal{F}\left[(-2\pi ix)^\alpha f\right]$

Therefore, we have, for each multi-indices α and β,

$$\begin{aligned}\rho_{\alpha,\beta}(\mathcal{F}[f]) &= \left\|\omega^\alpha \partial^\beta \mathcal{F}[f](\omega)\right\|_{L^\infty} = \left\|\omega^\alpha \mathcal{F}\left[(-2\pi ix)^\beta f\right]\right\|_{L^\infty} = \left\|\frac{(-2\pi i)^{|\beta|}}{(2\pi i)^{|\alpha|}} \mathcal{F}\left[\partial^\alpha x^\beta f\right]\right\|_{L^\infty} \\ &\leq (2\pi)^{|\beta|-|\alpha|} \left\|\partial^\alpha x^\beta f\right\|_{L^1} < \infty \qquad \text{(from the integral expression of } \mathcal{F})\end{aligned}$$

as desired. Here, the last L^1-integrability of $\partial^\alpha x^\beta f$ follows as the integrand is written as a finite sum of (polynomial) $\times\, \partial^\gamma f$ and this decreases faster than $(1 + |x|^{2n})^{-1}$, for example, which is integrable.

Therefore, we have shown that the image of $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$ is included in $\mathcal{S}(\mathbb{R}^n)$. As the Fourier inversion $\mathcal{F}^{-1}$ has almost the same expression as $\mathcal{F}$, we can similarly prove that this map is a self-bijection.

For the continuity with respect to $d_{\mathcal{S}}$, it suffices to prove that $\rho_{\alpha,\beta}(\mathcal{F}[f]) \to 0$ as $d_{\mathcal{S}}(f, 0) \to 0$ for each α, β as $d_{\mathcal{S}}$ is translation-invariant (the case of $\mathcal{F}^{-1}$ can also be done from this via variable transformation). We can prove this by refining the argument of proving $\rho_{\alpha,\beta}(\mathcal{F}[f]) < \infty$. Indeed, by writing $\partial^\alpha x^\beta f = \sum_{(\gamma,\delta)\in\Gamma} c_{\gamma,\delta} x^\gamma \partial^\delta f$, where $\Gamma = \Gamma(\alpha, \beta)$ is a finite set of pair of multi-indices and $c_{\gamma,\delta}$ are constants, we have

$$(1 + |x|^{2n})\partial^\alpha x^\beta f = \sum_{(\gamma,\delta)\in\Gamma} c_{\gamma,\delta}(1 + |x|^{2n})x^\gamma \partial^\delta f = \sum_{(\gamma,\delta)\in\Gamma} c_{\gamma,\delta} \sum_{\epsilon\in E} c'_\epsilon x^{\gamma+\epsilon} \partial^\delta f,$$

where E is a finite set of multi-indices such that $1 + |x|^{2n} = \sum_{\epsilon\in E} c'_\epsilon x^\epsilon$. Therefore,

$$\sup_{x\in\mathbb{R}^n} (1 + |x|^{2n})|\partial^\alpha x^\beta f| \leq \sum_{(\gamma,\delta,\epsilon)\in\Gamma\times E} c_{\gamma,\delta} c'_\epsilon \rho_{\gamma+\epsilon,\delta}(f)$$

holds, and so we finally obtain

$$\rho_{\alpha,\beta}(\mathcal{F}[f]) \leq (2\pi)^{|\beta|-|\alpha|} \left\|\partial^\alpha x^\beta f\right\|_{L^1} \leq \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}x}{1 + |x|^{2n}}\right) \sum_{(\gamma,\delta,\epsilon)\in\Gamma\times E} c_{\gamma,\delta} c'_\epsilon \rho_{\gamma+\epsilon,\delta}(f) \to 0$$

as $d_{\mathcal{S}}(f, 0)$ tends to zero. □

We next prove the relation of $\mathcal{S}(\mathbb{R}^n)$ with other common function spaces. Denote the space of all smooth (i.e., infinitely differentiable) functions with a compact support by $C_c(\mathbb{R}^n)$. $C_c(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is obvious.

Theorem 11. *$C_c(\mathbb{R}^n)$ is a dense subset of $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Take an arbitrary $f \in \mathcal{S}(\mathbb{R}^n)$. It suffices to prove that for each positive integer m and $\varepsilon > 0$, there exists a $g \in C_c(\mathbb{R}^n)$ such that $\rho_{\alpha,\beta}(f-g) < \varepsilon$ for all $|\alpha|, |\beta| \leq m$.

Take a function $h \in C_c(\mathbb{R}^n)$ such that $h(x) = 1$ on $|x| \leq 1$, $h(x) \in [0,1]$ on $|x| \in [1,2]$, and $h(x) = 0$ on $|x| \geq 2$. Such a function indeed exists; it can be constructed by exploiting the one-dimensional smooth function

$$t \mapsto \begin{cases} 0 & (t \leq 0) \\ e^{-1/t} & (t > 0) \end{cases},$$

but we omit the details here. For a positive integer N, define $h_N(x) := h(N^{-1}x)$. Then, for each multi-index α, $\partial^\alpha h_N(x) = N^{-\alpha}(\partial^\alpha h)(N^{-1}x)$ holds. In particular, we have

$$N \max_{0<|\alpha|\leq m} \sup_{x\in\mathbb{R}^n} |\partial^\alpha h_N(x)| \leq \max_{0<|\alpha|\leq m} \sup_{x\in\mathbb{R}^n} |\partial^\alpha h(x)| =: C < \infty \tag{7}$$

for each N. We shall prove that $\lim_{N\to\infty} \rho_{\alpha,\beta}(h_N f - f) = 0$ for all $|\alpha|, |\beta| \leq m$. Fix α and β such that $|\alpha|, |\beta| \leq m$. Then, from the Leibniz rule, we have

$$x^\alpha \partial^\beta (h_N f - f) = (h_N - 1) x^\alpha \partial^\beta f + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1 \neq 0}} c_{\beta_1,\beta_2} (\partial^{\beta_1} h_N)(x^\alpha \partial^{\beta_2} f)$$

for some positive integer constants c_{β_1,β_2}. By using (7), we obtain

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta (h_N f - f)| \leq \sup_{|x|\geq N} |x^\alpha \partial^\beta f(x)| + \frac{C}{N} \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1 \neq 0}} c_{\beta_1,\beta_2} \rho_{\alpha,\beta_2}(f).$$

The second term in the right-hand side obviously tends to zero, whereas the convergence first term is also clear from $\sup_{x\in\mathbb{R}^n} |x||x^\alpha \partial^\beta f(x)| < \infty$. Hence, the assertion of the theorem holds. □

From the usual density result of $C_c(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ (e.g., Theorem 1.5.8 in the lecture note), we can prove the following theorem.

Theorem 12. *For $1 \leq p < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $L^p(\mathbb{R}^n)$. Moreover, the inclusion map $\mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous.*

Proof. As $C_c(\mathbb{R}^n)$ is a dense subset of $L^p(\mathbb{R}^n)$, the former assertion is clear just from the inclusion $C_c(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

For the latter part, we follow the proof of Proposition 2.2.6 in Grafakos (2008). For an arbitrary $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} \|f\|_{L^p}^p &= \int_{|x|\leq 1} |f(x)|^p \, \mathrm{d}x + \int_{|x|\geq 1} \frac{1}{|x|^{n+1}} |x^{n+1}||f(x)|^p \, \mathrm{d}x \\ &\leq \sup_{x\in\mathbb{R}^n} |f(x)|^p + \left(\int_{|x|\geq 1} \frac{\mathrm{d}x}{|x|^{n+1}} \right) \sup_{x\in\mathbb{R}^n} \left| x^{\lceil \frac{n+1}{p} \rceil} f(x) \right|^p, \end{aligned}$$

and so it follows that $f_n \to f$ in $d_{\mathcal{S}}$ implies $f_n \to f$ in L^p. □

Let us go back to the space $H^{1/2}(\mathbb{R}^n)$.

Theorem 13. *$\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $H^{1/2}(\mathbb{R}^n)$. Moreover, the inclusion $\mathcal{S}(\mathbb{R}^n) \subset H^{1/2}(\mathbb{R}^n)$ is continuous.*

Proof. The inclusion follows from $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$ (Theorem 10). Indeed, as $\mathcal{F}[f] \in \mathcal{S}(\mathbb{R}^n)$ holds for each $f \in \mathcal{S}(\mathbb{R}^n)$, $(1+2\pi|\omega|)|\mathcal{F}[f](\omega)|^2$ is integrable.

We next prove the density. Take an arbitrary $f \in H^{1/2}(\mathbb{R}^n)$. Then, from the definition of $H^{1/2}(\mathbb{R}^n)$, $(1+4\pi^2|\omega|^2)^{1/4}\mathcal{F}[f](\omega)$ is in $L^2(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} (1+4\pi^2|\omega|^2)^{1/2}|\mathcal{F}[f](\omega)|^2 \,\mathrm{d}\omega \le \int_{\mathbb{R}^n} (1+2\pi|\omega|)|\mathcal{F}[f](\omega)|^2 \,\mathrm{d}\omega < \infty.$$

Hence, from Theorem 12 with $p = 2$, there exists a sequence $g_n \in \mathcal{S}(\mathbb{R}^n)$ which is convergent to $(1+4\pi^2|\omega|^2)^{1/4}\mathcal{F}[f]$ in L^2. As $(1+4\pi^2|\omega|^2)^{-1/4}$ is smooth, $(1+4\pi^2|\omega|^2)^{-1/4}g_n$ is also smooth. We prove this function is indeed in $\mathcal{S}(\mathbb{R}^n)$. For each multi-index α, we can prove that

$$\partial^\alpha (1+4\pi^2|\omega|^2)^{-1/4} = \sum_{i=0}^{|\alpha|} c_i(x)(1+4\pi^2|\omega|^2)^{-1/4-i}$$

for some polynomials c_i, by induction. Combining this with the Leibniz rule, we see that

$$\sup_{x\in\mathbb{R}^n} \left|\omega^\alpha \partial^\beta \left((1+4\pi^2|\omega|^2)^{-1/4} g_n\right)\right| < \infty$$

holds for each α and β. Therefore, by letting $f_n := \mathcal{F}^{-1}\left[(1+4\pi^2|\omega|^2)^{-1/4}g_n\right]$, we have

$$\|f_n - f\|_{H^{1/2}} \le 2^{1/4}\left\|(1+4\pi^2|\omega|^2)^{1/4}(\mathcal{F}[f_n]-\mathcal{F}[f])\right\|_{L^2} = \left\|g_n - (1+4\pi^2|\omega|^2)^{1/4}\mathcal{F}[f]\right\|_{L^2} \to 0$$

as $n \to \infty$. Since we know $f_n \in \mathcal{S}(\mathbb{R}^n)$ from Theorem 10, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{1/2}(\mathbb{R}^n)$.

Finally, we shall prove the continuity of the inclusion. This is done in a similar manner to the previous theorem. Indeed, we have, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\|f\|_{H^{1/2}}^2 &\le \int_{|\omega|\le 1}(1+2\pi)|\mathcal{F}[f](\omega)|^2 \,\mathrm{d}\omega + \int_{|\omega|\ge 1}\frac{1+2\pi}{|\omega|^{n+1}}|\omega|^{n+2}|\mathcal{F}[f](\omega)|^2 \,\mathrm{d}\omega \\ &\le (1+2\pi)\sup_{\omega\in\mathbb{R}^n}|\mathcal{F}[f](\omega)|^2 + \left(\int_{|\omega|\ge 1}\frac{1+2\pi}{|\omega|^{n+1}}\,\mathrm{d}\omega\right)\sup_{\omega\in\mathbb{R}^n}\left||\omega|^{\lceil\frac{n+2}{2}\rceil}\mathcal{F}[f](\omega)\right|^2.\end{aligned}$$

Thus, as $f \mapsto \mathcal{F}[f]$ is continuous on $(\mathcal{S}(\mathbb{R}^n), d_{\mathcal{S}})$, $\mathcal{S}(\mathbb{R}^n) \subset H^{1/2}(\mathbb{R}^n)$ is a continuous embedding. □

Combining those results, we finally obtain the following assertion.

Theorem 14. *$C_c(\mathbb{R}^n)$ is a dense subset of $H^{1/2}(\mathbb{R}^n)$.*

Proof. As $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{1/2}(\mathbb{R}^n)$ (Theorem 13), for each $\varepsilon > 0$ and $f \in H^{1/2}(\mathbb{R}^n)$, there is an $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g - f\|_{H^{1/2}} \le \varepsilon/2$. As $C_c(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ (Theorem 11), there exists a sequence $g_n \in C_c(\mathbb{R}^n)$ convergent to g under the metric $d_{\mathcal{S}}$. From the continuity of inclusion $\mathcal{S}(\mathbb{R}^n) \subset H^{1/2}(\mathbb{R}^n)$, $\|g_n - g\|_{H^{1/2}} \to 0$ also holds. Therefore, for a sufficiently large n, we have

$$\|g_n - f\|_{H^{1/2}} \le \|g_n - g\|_{H^{1/2}} + \|g - f\|_{H^{1/2}} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon.$$

As f and ε are arbitrary, the proof is complete. □

4 Sobolev inequality

In this final section, we try the generalization of the continuous embedding $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ (Theorem 2). Although the statement holds even for $p = 2n/(n-1)$ with $n \ge 2$ (Lieb & Loss 2001, Theorem 8.4; Di Nezza et al. 2012, Theorem 6.5), we here omit the proof for that case.

Theorem 15 (Sobolev inequality)**.** *For each $p \in [2, 2n/(n-1))$ (the right end is ∞ when $n = 1$), there exists a constant $C_{n,p} > 0$ satisfying*

$$\|f\|_{L^p(\mathbb{R}^n)} \le C_{n,p} \|f\|_{H^{1/2}(\mathbb{R}^n)}$$

for all $f \in H^{1/2}(\mathbb{R}^n)$. In particular, $H^{1/2}(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$.

Proof. This is the generalization of the proof of (Lieb & Loss 2001, Theorem 8.5). The case $p = 2$ has been already proven in Theorem 2, where we can take $C_2 = 1$. Let $f \in H^{1/2}(\mathbb{R}^n)$. Then, for $p > 2$, we take $q \in (1, 2)$ such that $1/p + 1/q = 1$. Then, we have, for an r such that $1/r + q/2 = 1$,

$$\begin{aligned} \|\mathcal{F}[f]\|_{L^q}^q &= \int_{\mathbb{R}^n} (1 + 2\pi|\omega|)^{-q/2} \left|\mathcal{F}[f](\omega)(1 + 2\pi|\omega|)^{1/2}\right|^q \mathrm{d}\omega \\ &\le \left\|(1 + 2\pi|\omega|)^{-q/2}\right\|_{L^r} \left\|\left|\mathcal{F}[f](\omega)(1 + 2\pi|\omega|)^{1/2}\right|^q\right\|_{L^{2/q}} \qquad \text{(from Hölder's inequality)} \\ &= \left\|(1 + 2\pi|\omega|)^{-q/2}\right\|_{L^r} \|f\|_{H^{1/2}}^q. \end{aligned}$$

From the Haussdorff-Young inequality (Lieb & Loss 2001, Theorem 5.7; Grafakos 2008, Proposition 2.2.16), we have $\|f\|_{L^p} \le \|\mathcal{F}[f]\|_{L^q}$, and so it suffices to confirm that $(1 + 2\pi|\omega|)^{-qr/2}$ is integrable.

Indeed, we have

$$r = \frac{1}{1 - q/2} = \frac{1}{1 - \frac{1}{2(1-1/p)}} = \frac{2(1 - 1/p)}{2(1 - 1/p) - 1} = \frac{2(p-1)}{p-2},$$

and so

$$\frac{qr}{2} = \frac{1}{1 - 1/p} \frac{p-1}{p-2} = \frac{p}{p-2}.$$

As we have

$$\int_{\mathbb{R}^n} (1 + 2\pi|\omega|)^{-p/(p-2)} \,\mathrm{d}\omega < \infty \tag{8}$$

when $p/(p-2) > n$, the conclusion holds. □

For $n \ge 2$, the end exponent $p = \frac{2n}{n-1}$ is called the *fractional critical exponent* (Di Nezza et al. 2012). In this proof, this exponent is indeed critical in whether or not achieving (8). Then, it is natural to consider the case $n = 1$ and $p = \infty$, which can be regarded as a critical case. However, there is a counterexample.

For each $N \ge 3$, consider the function

$$f_N(x) = \begin{cases} \frac{1}{x \log x} & (e \le x \le N) \\ 0 & (\text{otherwise}) \end{cases}.$$

Then, we have $\|\mathcal{F}^{-1}[f_N]\|_{L^\infty} \le \|f_N\|_{L^1}$, and this equality indeed holds as $\mathcal{F}^{-1}[f_N]$ is continuous due to $f_N \in L^1(\mathbb{R})$ and

$$\mathcal{F}^{-1}[f_N](0) = \int_{\mathbb{R}} f_N(x) \,\mathrm{d}x = \|f_N\|_{L^1}.$$

We can explicitly calculate as

$$\left\|\mathcal{F}^{-1}[f_N]\right\|_{L^\infty} = \int_e^N \frac{\mathrm{d}x}{x \log x} = \log\log N - \log\log e = \log\log N.$$

We also have

$$\left\|\mathcal{F}^{-1}[f_N]\right\|_{H^{1/2}}^2 \le (1 + 2\pi) \int_e^N \frac{\mathrm{d}x}{x(\log x)^2} = (1 + 2\pi)\left(1 - \frac{1}{\log N}\right).$$

Therefore, we obtain

$$\frac{\left\|\mathcal{F}^{-1}[f_N]\right\|_{L^\infty}}{\|\mathcal{F}^{-1}[f_N]\|_{H^{1/2}}} \ge (1 + 2\pi)^{-1/2} \left(1 - \frac{1}{\log N}\right)^{1/2} \log\log N \to \infty$$

as $N \to \infty$. In particular, there is no Sobolev inequality like $\|\cdot\|_{L^\infty} \le C_{1,\infty} \|\cdot\|_{H^{1/2}}$.

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